The Mathematics of Mathematics

Thinking with the Late, Spinozist Vygotsky

Wolff-Michael Roth

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NEW DIRECTIONS IN MATHEMATICS AND SCIENCE EDUCATION

Volume 32

Scope

Mathematics and science education are in a state of change. Received models of teaching, curriculum, and researching in the two fields are adopting and developing new ways of thinking about how people of all ages know, learn, and develop. The recent literature in both fields includes contributions focusing on issues and using theoretical frames that were unthinkable a decade ago. For example, we see an increase in the use of conceptual and methodological tools from anthropology and semiotics to understand how different forms of knowledge are interconnected, how students learn, how textbooks are written, etcetera. Science and mathematics educators also have turned to issues such as identity and emotion as salient to the way in which people of all ages display and develop knowledge and skills. And they use dialectical or phenomenological approaches to answer ever arising questions about learning and development in science and mathematics.

The purpose of this series is to encourage the publication of books that are close to the cutting edge of both fields. The series aims at becoming a leader in providing refreshing and bold new work—rather than out-of-date reproductions of past states of the art—shaping both fields more than reproducing them, thereby closing the traditional gap that exists between journal articles and books in terms of their salience about what is new. The series is intended not only to foster books concerned with knowing, learning, and teaching in school but also with doing and learning mathematics and science across the whole lifespan (e.g., science in kindergarten; mathematics at work); and it is to be a vehicle for publishing books that fall between the two domains—such as when scientists learn about graphs and graphing as part of their work.

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All chapters in this book have undergone peer review.

Printed on acid-free paper

Contents

Preface

One psychologist more than any other has influenced theories of learning and development: Lev S. Vygotsky. However, recent publications suggest that much of the way in which Vygotsky's work has been taken up does injustice to what the psychologist had actually been writing – to the point of totally misrepresenting the work.[1] During the first decade of the 21st century, Vygotsky's descendants have given Ekaterina Zavershneva access to the family archive.[2] In the process, she uncovered a wealth of unpublished and unheard-of private notes. In these notes that were written near the end of his life, Vygotsky expresses discontent with his own theory, the one most people who read Vygotsky think they are familiar with; and he deems them insufficient, requiring a complete overhaul and revision. In particular, although he had spent much of his scholarly life critiquing and attempting to overcome the Cartesian dualism that is characteristic of psychology then as now he had failed. In the notes, he acknowledges the remnants of Cartesian dualism in his work, including an over-emphasis of the intellectual over affect and the practical. The Cartesianism also characterizes current theoretical approaches, especially in (radical, social) constructivism; but, as philosophers have shown, the specters of Cartesianism exist even in embodiment and enactivist theory.[3]

To overcome these remnants in his own work, Vygotsky turned to the philosopher Baruch Spinoza, who had postulated that there was only *one substance* that has body (Extension) and Thought as attributes. Accordingly, there are not two substances, body and thought (mind), biology and culture, or nature and nurture, but only one substance that manifests itself in different, mutually exclusive ways.

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[1] See, e.g., Felix T. Mikhailov, 'The "Other Within" for the Psychologist', *Journal of Russian and East European Psychology* 39 no. 1 (2001): 6–31; and Anton Yasnitsky and René van der Veer, eds., *Revisionist Revolution in Vygotsky Studies* (London: Routledge, 2016).

[2] See, for example, Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive (1912–1934): New Findings', *Journal of Russian and East European Psychology* 48(1): 14–33; and Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive: New Findings – Notebooks, Notes, and Scientific Journals of L.S. Vygotsky (1912–1934)', *Journal of Russian and East European Psychology* 48(1): 34–60.

[3] See Maxine Sheets-Johnstone, *The Corporeal Turn: An Interdisciplinary Reader* (Exeter: Imprint Academic, 2009).

This one substance is the thinking body. This thinking body, however, is not the material human body: 'Thought can … only be understood through investigation of its mode of action in the system thinking body–nature as a whole'.[4] Because Vygotsky realized that there are some problems in Spinoza's work, he envisioned reading Spinoza through a Marxian lens generally and through the lens of *The German Ideology*[5] specifically, a book that was published for the first time in Moscow during the final 18 months of Vygotsky's life.

The most fundamental idea in the *Ideology* is the primacy of societal relations to anything that distinguishes humans from other species. Consciousness, the ideal, and the general all are societal in nature – not just as per their origin but also in their very existence. The ideal, such as a mathematical 'abstraction' or a mathematical 'idea', exists in the form of human relations. This insight led me to the title of this book, *The Mathematics of Mathematics*. That is, mathematics is not 'socially constructed' because humans have produced with others (i.e. 'socially') and 'negotiated' some idea. In this way of approaching mathematics, 'the social' is incidental. In any event, any mathematical discovery is ascribed to individuals and, thus, is understood not socially: something like Fermat's Last Theorem is considered to have been an individual construction before it became social. This, however, is not consistent with how Marx or the late Vygotsky thought, where *any higher psychological function* was *a relation with another person*. Therefore, what is mathematical in mathematics is not merely (contingently) social but exists in the form of (universal) societal relations. Children learn mathematics and mathematical forms because they exist in public, that is, because these exist *as* relations in which the children are integral part. Because, according to the *Ideology*, consciousness [Bewußtsein] is conscious being [bewußtes Sein], mathematics becomes individual when the child becomes conscious of the relation with others where the mathematical form exists as joint praxis.

Unfortunately, Vygotsky never got to make these revisions. In his final note prior to entering the hospital where he died only 30 days later, he likened himself to Moses, who had seen the Promised Land but was never allowed to enter it.[6] What was it that Vygotsky saw and wanted to develop for psychology? More importantly, how would research methods and findings look like if he had actually been able to do what he envisioned? What would theories of mathematical thinking look like if we were to follow the lines of thinking apparent from the notes of the late Vygotsky (i.e., during the last 18 months of his life)? This book is designed to show where such a revision takes us in mathematics education.

The revisions that Vygotsky planned included aspects of the theory that continue to be major topics of mathematics education. His ideas about the revision to be undertaken have radical consequences for the way in which mathematics educators think and theorize pertinent issues. Some of the shifts arising from thinking with the Spinozist-Marxian Vygotsky include: (a) mediational nature of the sign (lan-

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[4] Evald V. Il'enkov, *Dialectical Logic: Essays on its History and Theory* (Moscow: Progress Publishers, 1977), 52.

[5] Karl Marx, and Friedrich Engels, *Werke Band 3* (Berlin: Dietz, 1978).

[6] Ekaterina Iu. Zavershneva, 'Notebooks', 58.

guage) → semiotic (sense-giving) speech field, (b) meaning → sense, (c) the zone of proximal development → the primacy of the social, (d) thought → unity/identity of intellect, affect, and praxis, (e) thinking → thinking and speaking as two lines of development in communication, and (f) the distinction between intra- and intersubjectivity (inside–outside) → intra-intersubjectivity. These shifts involve a radical revision of what have come to be accepted as fundamental truths in the field of mathematics education, including the process of mediation (L. Radford), the establishment of socio-mathematical norms (P. Cobb), and the constructivist distinction between intrasubjectivity and intersubjectivity (T. Brown).

In this book, I take as a starting point Vygotsky's notes, writings, and presentations from the last 18–24 months of his life; from this starting point, I offer up a theoretical approach to mathematical thinking and learning that differs in some essential ways from other current theories. Episodes mostly from elementary school mathematics classrooms but some also involving scientists are used for developing and exemplifying the theoretical advances and the method of investigation. Published analyses of main contributors in mathematics education are used to show where existing work falls short with respect to meeting the requirements that arise from Vygotsky's revision of his own theory.

A key problem with many English versions of Vygotsky's books and texts has been the problematic nature of the translations, which, in some instances, completely falsified what the scholar actually had written or said. Thus, for example, in many translations, the Russian equivalent of psychic [psixičeskij], has been rendered by means of 'mental' rather than, as Aleksander Luria suggested to Michael Cole, as 'psychological'.[7] The Russian language has two terms that tend to be used where English might employ mental – mental'nyj and duxovnyj. The term 'mental' for psixičeskij is inappropriate because the 'psychic' or 'psychological' includes more than the intellectual-mental. Translating psixičeskij as 'mental' is in direct contradiction with Vygotsky's attempt to do *unit* analysis as opposed to analysis by elements. To deal with this issue, I have in many instances replaced a term in a quotation, after consulting the original text, by means of a more appropriate word and indicate this replacement by means of square brackets. Thus, for example, 'higher [psychological] functions' is used instead of 'higher mental functions' when Vygotsky wrote, in Russian, 'psixičeskie funkcii'.

There are similar issues with the translations of Spinoza's Latin *Ethica* into English. Thus, Spinoza's general definition of the emotions[8] begins in one translation with the words 'The emotion called a passive experience is a confused idea whereby the mind affirms a greater or less force of existence of its body, or part of its body, than was previously the case'[9], whereas another translation renders the

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[7] Michael Cole, personal email, June 10, 2015.

[8] The Latin text reads, 'Affectus, qui animi pathema dicitur, est confuse idea, qua mens maiorem sui corporis vel alicuius eius partis existendi vim, quam antea, affirmat …'. See the end of part III of Benedicti de Spinoza, *Ethica: Ordine Geometrico Demonstrata et in Quinque Partes Distincta in Quibus Agitur*, accessed June 29, 2016, http://users.telenet.be/rwmeijer/spinoza/works.htm.

[9] Baruch Spinoza, 'Ethics', in *Complete Works,* trans. Samuel Shirley (Indianapolis: Hackett Publishing, 2002), 319.

same Latin text as: 'Emotion, which is called a passivity of the soul, is a confused idea, whereby the mind affirms concerning its body, or any part thereof, a force for existence [*existendi vis*] greater or less than before'[10]. Having studied seven years of Latin in high school allowed me to check the translations against the original. In each pertinent case, I then chose the one that I thought better or more readably represented the Latin original.

In this book, I use *scholarly* Romanization. Thus, when using Russian words, the scholarly, scientific transliteration produced by the *International Organization for Standardization* was used for the Romanization rather than the transliterations of the American Library Association or the British Standard – as this tends to be done in linguistics and philosophy. I therefore transliterate *pereživanie* [experience] and *obučenie* [teaching | learning], as linguists would do it, rather than perezhivanie and obuchenie, as many educators practice it.

Victoria, BC
December 2016

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[10] Benedict de Spinoza, *The Ethics (Ethica Ordine Geomtetrico Demonstrata*), trans. R. H. M. Elwes, available at http://www.gutenberg.org/files/3800/3800-h/3800-h.htm (Project Guttenberg EBook).

1

Vygotsky's Marxist-Spinozist Re/Orientations

> 'This is the final thing I have done in psychology – and I will die at the summit like Moses, having glimpsed the prom[ised] land but without setting foot on it. Farewell, dear creations'.[1]

Late in his life, Vygotsky became highly critical of his own previous work and began to (a) radically reorient his inquiries and, in the process, (b) radically rethink his already continuously developing theory. But his rethinking was only in the beginning when death ended his life. In the introductory quotation, written only weeks prior to his death, Vygotsky likens himself to Moses, who had seen the Promised Land but never could actually reached and set foot on it. What was Vygotsky seeing? More importantly, he was in the process of radically revising his existing work, which means, he is celebrated for work today that he was actually in the process of completely overturning. One entry in his personal notebooks, dated to some time between 1931 and 1933, reads like a programmatic instruction to himself: 'Bring Spinozism to life in Marxist psychology'.[2] Indications of where the thoughts occurring to him were leading are apparent from his personal notes and the final pieces of writing that were published only posthumously – including the first chapter and the final paragraphs of *Thinking and Speech*.[3] These textual pieces contain several quotations and paraphrases from Marx and Engel's *The German Ideology*, a work that was published for the first time in Moscow in 1932 (German) and 1933 (Russian).[4] In this chapter, based on the published personal notes and the

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[1] Lev S. Vygotsky, quoted in Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive: New Findings – Notebooks, Notes, and Scientific Journals of L.S. Vygotsky (1912–1934)', *Journal of Russian and East European Psychology* 48(1), 58.

[2] Lev S. Vygotsky, 'Two fragments of personal notes by L. S. Vygotsky from the Vygotsky family archive', *Journal of Russian and East European Psychology* 48(1) (2010), 93.

[3] Lev S. Vygotsky, 'Thinking and Speech', in *The Collected Works of L. S. Vygotsky. Vol 1: Problems of General Psychology* (New York: Springer, 1987).

[4] Karl Marx, and Friedrich Engels, *Werke Band 3* (Berlin: Dietz, 1978). The quotation marks were removed from the original publication of *Thinking and Speech*, first in subsequent Russian editions, later in the English translations that were not based on the original but on the subsequent, changed versions; there were also parts of the text removed. In the quotations used here, the original of *Thinking and Speech* was used including the omitted text and quotation marks.

last texts and lectures Vygotsky produced, his radically revised orientation to thinking and learning mathematics is described.

Spinoza

> 'Mind and body – are one and the same individual thing, conceived now under the attribute of Thought and now under the attribute of Extension'.[5]

Baruch Spinoza holds a special position not only in the later Vygotsky's thinking but also in the philosophical tradition of materialist dialectics – or dialectical materialism – more generally. Already the German idealist philosopher Georg Wilhelm Friedrich Hegel noted that 'Spinoza is the main viewpoint of modern philosophy: either Spinozism or no philosophy'.[6] Evald V. Il'enkov, a philosopher of materialist dialectics, would later refer to the same Hegel work to emphasize the influence Spinoza had on the subsequent development of dialectical thought. Il'enkov, however, identifies one problem that limits the generality of Spinoza's own work, which exists in the non-coincidence of the logic of Spinoza's thinking with the formal logic underlying the movement in which he developed his system (axioms, theorems, scholia, and proofs).

Foundational Determinations

In contradistinction to René Descartes, whose ontology begins with two basic, incommensurable kinds of substances, body (extension) and soul (mind), Baruch Spinoza begins his unified theory with the definition of *substance* as something conceived only through itself.[7] As a result, an extended body can determine another body, just as thought can influence thought. But no relation is possible across kinds so that the extended body cannot influence thought, just as thought cannot causally affect the body.[8] If body (extension) and thought are related then it is because there is a substance much more original than each and transcending both: 'substance is by nature prior to its affections'.[9] There is one substance, which constitutes Nature as a whole. But this substance cannot be grasped or seen as such, because in thinking, it is reduced to thought; and in perception, it is reduced to

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[5] Baruch Spinoza, 'Ethics', in *Complete Works* (Indianapolis, IN: Hackett Publishing, 2002), 259.

[6] Georg Wilhelm Friedrich Hegel, *Werke Band 20* (Frankfurt/M: Suhrkamp, 1979), 162–163.

[7] Spinoza, 'Ethics', 217ff.

[8] This is one of the key points of Lucy Suchman's work on plans and situated actions. Plans do not cause situated actions but rather orient it in a general way. After the fact, they describe, more or less well, whether a person has done what the plans foresaw. See Lucy Suchman, *Human-Machine Reconfigurations: Plans and Situated Actions* (Cambridge: Cambridge University Press, 2007).

[9] Spinoza, 'Ethics', 218.

extension. Extension and thought are attribut*e*s of the substance, which manifest themselves in the modes of body and idea, the two affectations of substance of interest. Each mode is something else than the substance, and thus again are substances external to each other that must be conceived through themselves. They are therefore, unable to communicate with each other once conceived in separate ways. This will become the key differences between other approaches seeking to integrate body and mind – see chapter 2 – and the Spinozist-Marxian take that Vygotsky was beginning to articulate in his notes and field-charting texts for the projects to come on affect and consciousness. *Existence*, being, is part of the nature of substance and, therefore, cannot be the distinguishing feature between the different modes.

Substance cannot be a (finite) material thing because it 'would not be substance, if it were an object of experience, i.e., something perceivable; for then it would be a determined, special thing, i.e., a sensual, which is only a finite affection of substance, but not substance'.[10] To understand the relationship between substance and its modes, it is useful to explicate a scientific method articulated only in the 20th century: the documentary method. It is used in the social sciences and humanities in the case of abstract, directly inaccessible phenomena, such as 'worldview' or 'zeitgeist'. None of the associate phenomena is directly accessible to observation. Instead, the phenomenon manifests itself – and is perceived – in documentary evidence. Thus, for example, a painting by Paul Cezanne would be a document of the period 'impressionism' in the field of the fine arts much in the same way as a composition by Claude Debussy would be a document of the equivalent period in music. That is, *impressionism* is not accessible directly but only indirectly, through the different ways that it manifests itself. An analogy may assist in the contradictory ways in which something may appear are ambiguous or bi-stable forms, such as the drawing that appears as duck or rabbit (Fig. 1.1). Each is a manifestation of a whole, a duck–rabbit.

In the same way, the Spinozist *substance* is not accessible other than through its modes. But substance does not merely exist in thought – it could not manifest itself in material form if it were only of the thinking type. It is therefore incomparable, as it does not have an equal. Its reality exists in and of itself. In it, the difference between being and thinking, body (Extension) and mind (Thought), is dissolved. As soon as we begin to think substance *as* thinking/thought or as material, then we have lost the very thing we have started out with. Thus, just as 'light discloses itself as light, substance discloses itself as substance'.[11] Just as it is irrational to ask whether the light I see truly is light rather than darkness, or whether it is particle or wave, the denial of substance would not just take something away from substance but would negate it completely.

This position has consequences for approaching the results of neuropsychological studies, where the relationship of thinking and brain is investigated by means of

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[10] Ludwig Feuerbach, 'Spinoza' in *Geschichte der neuern Philosophie von Bacon von Verulam bis Benedict Spinoza* (Ansbach: C. Brügel, 1833), 373.
[11] Feuerbach, 'Spinoza', 376.

Fig. 1.1 The drawing can be seen as a duck or as a rabbit, but not both at the same time. One may therefore ask, 'Is it a duck or is it a rabbit?' Neither explains itself by the other. A way out of the quandary is to assume some phenomenon that manifests itself in contradictory ways. 'Duck' and 'rabbit' are the analogy of the two modes of one and the same phenomenon (substance)
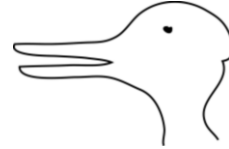
functional magnetic resonance imaging (fMRI) studies. While the human subject does some task, picking up a cup of coffee, looking at images, or listening to someone else's talk, an image is made of the activity in the brain. These recordings, therefore, represent the biological manifestation of the activity, whereas thinking (understanding, consciousness) is its counterpart in the mode of thought. From the Spinozist position, it is 'impossible either to understand thought through an examination, however exact and thorough, of the spatially geometric changes in the form of which it is expressed within the body of the brain, or, on the contrary, to understand the spatial, geometric changes in the brain tissue from the most detailed consideration of the composition of the ideas existing in the brain'.[12] This is so because, as Spinoza insists, the two forms of movement (change) are the different manifestations of one and the same phenomenon. That phenomenon, thereby, falls outside of the description. Hegel would later launch a very similar critique at the phrenologists of his time, who attempted to correlate bumps on the skull with consciousness of the person.[13] In the same vein, Vygotsky would note that 'it is ridiculous to look for specific centers of higher psychological functions or supreme functions in the cortex … they must be explained not on the basis of *internal organic* relations … but in external terms'.[14]

When two things have nothing in common, such as the two modes of body (extension) and thought, then one cannot be the *cause* of the other. This third proposition of the *Ethics* is counter to almost everything that we are familiar with in current research, for it means that thought cannot cause bodily action, including the bodily movements that bring about speech, and material action cannot cause thought. Some readers may be aware of the research that exhibits the gap between plans, forms of thought, and situated material action[15] – a confirmation of the suitability of the Spinozist conceptualization.

Differing from other ways of conceiving cause and effect, where cause is external to the thing (mode) affected, the Spinozist-Marxian approach takes cause to be *immanent* rather than *transitive* to things. Characteristic of the former way of thinking exists when teachers, what they say or do, somehow are considered to be responsible in one or another way for student learning. Such ways would include

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[12] Evald V. Il'enkov, *Dialectical Logic: Essays on its History and Theory* (Moscow: Progress Publishers, 1977), 36.

[13] Georg Wilhelm Friedrich Hegel, *System der Wissenschaft, Erster Theil, die Phänomenologie des Gesistes* [System of Science, Part I, Phenomenology of Spirit] (Bamberg und Würzburg: J. A. Goebhardt, 1807), 259–286.

[14] Lev S. Vygotsky, 'Concrete Human Psychology', *Soviet Psychology* 27 no. 2 (1989), 59.

[15] See, for instance, Suchman, *Reconfigurations* and Wolff-Michael Roth, 'Radical Uncertainty in Scientific Discovery Work', *Science, Technology & Human Values* 34 (2009): 313–336.
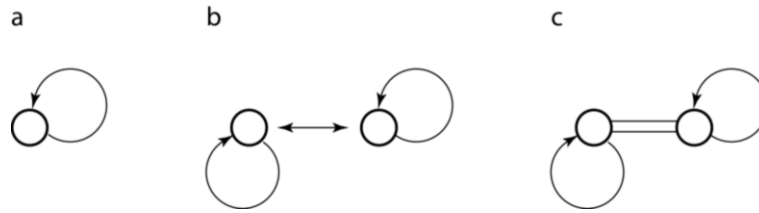
a   b   c

Fig 1.2 **a** In the self-actional model, the Self constructs itself, its knowledge being tested for viability through actions in world forever inaccessible. **b** In the *inter*actional model, two (or more) autonomous Selves relate to each other, each external to the other. **c** In the transactional model, the Selves cannot be specified independently: they are interdependent parts of a whole, each reflecting the whole and, therefore, also the other

thinking of teacher activity as 'scaffolding' students' construction of 'meaning'. Any action inherently is material, so that it does not make sense in a Spinozist take to say that students' thinking changes as a consequence of teacher actions. Instead, the Spinozist take is consistent with transactional approaches, such as those that are articulated by John Dewey or Gregory Bateson. The transactional perspective is opposed to the self-actional and *inter*actional models.[16] In self-actional models of knowing, the individual subject is coiled upon itself, constructing within its mind its own intrasubjective structures (Fig. 1.2a). It is an informationally closed system, which, at best, can test its constructions with its experience in the world. In *inter*actional models, two or more independent Selves (monads) come together to construct the objects of the world and knowledge together, which each then constructs (internally) for itself (Fig. 1.2b). Transactional models are inherently Spinozist, for the philosopher recognized that 'we are passive insofar we are a part of Nature which cannot be conceived independently of other parts'.[17] Thus, any part of the system is given in terms of the system as a whole, that is, it is given in terms of all other parts. In the relation of two Selves, the whole relation determines these Selves, as much the Selves constitute the relation (Fig. 1.2c). For scholars such as Bateson and Dewey, the system investigated includes observers such that it is impossible to decouple observer and observed.

Thought is an attribute of the originary one substance, Nature (God), which, in this, becomes a 'thinking thing'. The two opposites, mind and matter (body) are one in substance. In other words, the notion of substance is present in the contradictory notions and, therefore, is not sublated (abolished). Instead, it is independent of the two such that it can be one or the other, and, in fact, it is both. Substance,

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[16] Dewey and Bentley coined the terms self-actional, interactional, and transactional. In a review of the literature on situated cognition, Roth and Jornet used these adjectives to distinguish different models of situated cognition as these have been proposed in different fields concerned with knowing. See John Dewey, and Arthur F. Bentley, 'Knowing and the Known' in *Useful Procedures of Inquiry* edited by Rollo Handy and E. C. Harwood (Great Barrington: Behavioral Research Council, 1999); and Wolff-Michael Roth, and Alfredo Jornet, 'Situated Cognition', *WIREs Cognitive Science* 4 (2013): 463–478.

[17] Spinoza, 'Ethics', 324.

therefore, is a singular plural or, which is the same, a plural singular. At the same time, because any mode has to be considered within one or the other attribute, thought and matter can be distinguished and separated from the thinking thing.

Spinoza notes in Proposition 7 of the *Ethics* part II: '*The order and connection of ideas is the same as the order and connection of things*'.[18] Substance may be considered under the attribute of thinking or under the attribute of extension, which leads us to thinking substance and extended substance. But the two are the same substance, simply considered under the lights of different attributes. Thus, whether Nature is considered 'under the attribute of Extension or under the attribute of Thought or under any other attribute, we find one and the same order, or one and the same connection of causes'.[19] This notion is fundamental to the Spinozist-Marxian approach, where categories are to be developed that indeed correspond to the phenomenon. A psychology based on this approach therefore pursues the genetic, historical reconstruction of its fundamental categories such that these also account for the psychological phenomena and how they are thought about during each epoch. Such historically developed categories may be, and in many cases are, inadequate and confused because they also have an internal logic, as do adequate, clear, and distinct ideas. The historical reconstruction therefore accounts for both the development of real phenomena – i.e. part of the extended substance – and the development of thinking about them – i.e. part of the thinking substance. The reconstruction is required because other approaches – e.g. classical psychology or sociology – operate with everyday concepts that are operationalized within science (see below, section on method). Few scholars seem to be attuned to the fact that the '*the preconstructed is everywhere*' so that the 'first and most pressing scientific priority … would be *to take as one's object the social work of construction of the pre-constructed object*'.[20] This is precisely the point of historically reconstructing psychological categories.

Thought is an attribute of Nature. Its infinite mode is the infinite intellect, whereas its finite mode is the individual mind. The individual human mind therefore is part of the infinite intellect. When there are multiple ideas – concretely realized in different individual minds – then each mind considers the thing partially, one-sidedly, and, thus, inadequately. The 'idea of the body' and 'the body' are one and the same individual thing, considered under the attribute Thought and Extension, respectively. But each of these two attributes, which can be considered only in their own terms, expresses substance only in its own ways, as a particular type: each manifests substance only one-sidedly.

Spinoza proves the proposition that humans exist in the modes of mind and body, united in the person. There is a correlation of the capacity to conceive of things and the variety of states of the body. Some scholars interpret this correspondence as constituting a parallelism of two different, autonomous things, au-

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[18] Spinoza, 'Ethics', 247.
[19] Spinoza, 'Ethics', 247.
[20] Pierre Bourdieu, 'The Practice of Reflexive Sociology (The Paris Workshop)', in *An Invitation to Reflexive Sociology* (Chicago: University of Chicago Press, 1992), 235 and 229.

tonomous, because each has to be conceived through itself.[21] The correspondence exists because body and mind are modes of the same single substance, the order of which is expressed identically in each mode. The Spinozist-Marxian take differs from others in how mind and body are approached, that is, through the unity/identity of one substance, which can be understood only through the consideration of that substance, versus through the parallelism of two autonomous systems between which there cannot be any real action.

To explain the relationship between thinking (ideas) and the world, Spinoza develops (a) a list of propositions about the relationship between bodies generally before getting to the relationship between bodies external to the thinking human body and the thinking human body itself and (b) propositions about the relationship between affections of the human body and ideas of these affections. He then proves Proposition II.39: 'Of that which is common and proper to the human body and to any external bodies by which the human body is customarily affected, and which is equally in the part as well as in the whole of any of these bodies, the idea also in the mind will be adequate'.[22] The human thinking body, in contrast to non-thinking bodies, evolves the shape (i.e. trajectory) of its movement in space such that it will conform to the shape, configuration, and position of that other body. That is, the thinking body coordinates the shape of its own movement with the shape of the other body. Take the case of the hands of an expert typist, whose hand and finger movements are such that s/he no longer needs to look at the keyboard to get on paper or monitor whatever text s/he wants. The hand and finger movements have been shaped by the layout of the keyboard. Now we might think that the movement of a compass and a circle also are coordinated, but this is so because of the way in which the compass is constructed. Compared to the compass, the human thinking body builds its shape on many differently formed bodies. It is precisely this feature that distinguishes the thinking body – e.g. of humans – and the non-thinking body – e.g. the compass. This Spinozist definition allows other bodies to be thinking bodies (e.g. animals), though differently and perhaps more limited, such as that of chimpanzees that are capable of making and using (simple) tools.

An interesting aspect in the system of Spinoza is the position on the question of the truth and correctness of ideas and the distinction from error. Because '*the human mind has no knowledge of the body, nor does it know it to exist, except through ideas of the affections by which the body is affected*', '*there is nothing positive in ideas whereby they can be said to be false*'; that is, 'inadequate and confused ideas follow by the same necessity as adequate, or clear and distinct ideas'.[23] Falsity arises from the fragmentary nature of ideas. There are consequently no inadequate or confused ideas unless these are the ideas of a specific individual. True and false ideas are distinguishable based on 'reason' and 'intuition' (i.e. induction and abduction in modern parlance), which are the forms of knowledge of the second and third kind Spinoza identifies; and only knowledge of the first kind,

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[21] See, e.g., the discussion of the body–mind (parallelism) by Gilles Deleuze, *Spinoza: Practical Philosophy* (San Francisco, CA: City Light Books, 1988), 86–91.

[22] Spinoza, 'Ethics', 265.

[23] Spinoza, 'Ethics', 258, 263, 264.

knowledge from casual experience and symbols, is the origin of falsity. In this Spi-
nozist take, therefore, 'the erring man also acted in strict accordance with a thing's
form, but the question was what the thing was'.[24] This aspect of the theory should
lead us to reconsider children's ideas in mathematics, which are not because this or
that child thinks incorrectly, but because they are integral aspect of the symbols
they find in their environment. The errors need to be situated in the cultural context
rather than sought in the (incorrect wiring, structures of the) mind.

Implicit in the Spinozist position is the fact that the body, considered as a body,
cannot have (mental) schemas of its own (future) actions that are somehow located
within the body generally and the mind specifically. For (the movement of) think-
ing to be appropriate, the only requirement that needs to be fulfilled is that of the
thinking body to act in accordance with any present situation, its forms and ar-
rangements. Thus, '*the idea of any mode wherein the human body is affected by
external bodies must involve the nature of the human body together with the nature
of the external body*'.[25] This, indeed, can be considered to be the function that was
selected during the evolution of the human species because it allowed the organism
to act appropriately according to the present requirements. In this situation, there-
fore, the structure of the thinking body, including that of the brain, tells us little
about what the thinking body actually does. This is so because the condition to
which it adapts, the 'cause' of its action, generally lies outside of the thinking
body. From a Spinozist position it is therefore 'necessary to elucidate and discover
in the thinking thing those very structural features that enable it to perform its spe-
cific function, i.e. to act, not according to the scheme of its own structure but ac-
cording to the scheme and location of all other things, including its own body'.[26]
That is, this position requires us to investigate the real system, thinking body in its
context within which it does and thinks, qua organ of doing and thinking. 'Thought
can therefore only be understood through investigation of its mode of action in the
system thinking body–nature as a whole'.[27] Readers certainly recognize this as the
underpinning of the entire research field known under the label 'situated cogni-
tion'. For Spinoza, then, thought is not found in any single body, or system of bod-
ies, but only in Nature considered in its entirety. *Nature* 'is the efficient cause of all
things that can come within the scope of the infinite intellect', generating, among
others, partial forms of itself realized in the human mind [that] perceives a thing
partially and inadequately'.[28]

The thinking body creates an adequate idea of the forms of the objects by creat-
ing an adequate idea of itself, that is to say, by creating an adequate idea of the
form of its own movement along the surface of external objects. As shown in the
preceding typewriter example *the movements of the hands have taken the form of
the keyboard, they have the same contour*. Research on eye movements during
perception shows that the eyes are indeed 'following' the outline of the object (Fig.

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[24] Il'enkov, *Dialectical Logic*, 58.
[25] Spinoza, 'Ethics', 256.
[26] Il'enkov, *Dialectical Logic*, 51.
[27] Il'enkov, *Dialectical Logic*, 52.
[28] Spinoza, 'Ethics', 227, 250.

Fig. 1.3 The Spinozist take is consistent with research on eye movement. Our eyes are in constant movement, and it is this movement that allows us to see and know anything at all. If a person is asked to count the number of straight lines in a drawing that includes the set in (a), research has shown that the eyes move in ways similar to (b), i.e., by 'following' each line through appropriate fixations
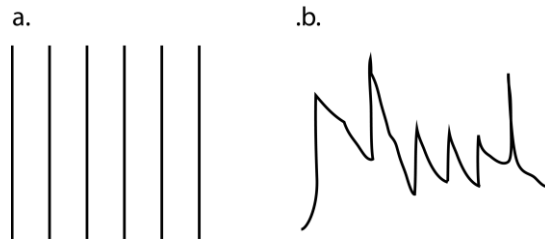
1.3). It is from the *movement* of the eyes 'around' the object that we come to know. Relatively recent neuroscientific research shows that objects and space are cognized by means of motor neurons that are active when the body moves. That is, we do not see because the world is represented on the retina in the way a photograph is made on a photographic medium – traditional film or electronic charge-coupled device in digital cameras – but because 'movements carve progressively out working space from undifferentiated visual information'.[29] Watching a video clip of someone being disgusted activates similar parts of the brain as when the same person smells a disgusting odor.

But if we recognize a «line» (the material thing) as 'a line' (the 'concept') because our eyes move in a particular way, then the work of my eyes is like that of yours. The same things have shaped their movements in the same way. The social in perception, therefore, arises from the things, which give themselves to all of us and thereby shape the movements of our senses, *not from our individual constructions*!

From the Spinozist position, it is 'in possessing consciousness of my own state (actions along the shape of some contour or other), [that] I thus also possess a quite exact awareness (adequate idea) of the shape of the external body'.[30] Therefore, there is a positive relation between the number of actions a body can perform and its thinking. Spinoza concludes that 'the mind is more capable of perceiving more things adequately in proportion as its body has more things in common with other bodies'.[31] The active aspect of perception also is apparent in the experiences with perceptual puzzles, as the duck–rabbit above (Fig. 1.1), where we might see a drawing initially only under one aspect (perceive one attribute) but where, once shown or encouraged to see something else, suddenly another aspect appears. From that moment on, we will have no trouble seeing one or the other aspect – the eyes have learned to move such that one or the other aspect appears in our conscious perception.

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[29] Giacomo Rizzolatti, Luciano Fadiga, Leonardo Fogassi, and Vittorio Gallese, 'The Space Around Us', *Science* 277 (1997): 190–191.

[30] Il'enkov, *Dialectical Logic*, 70.

[31] Spinoza, 'Ethics', 266.

Research in science classrooms on the birth of signs is consistent with rather than contradicting this take. Thus, students asked to explain the results of their investigations initially reproduced the movements required to do the investigations, then used the same hand/arm movements but symbolically, before eventually developing verbal descriptions that took over from and did the communicative work of the earlier movements.[32]

This way of approaching the issue offers opportunities to overcome Cartesian dualism, because thinking arises from acting along the contours of the social and material world. The actions of the thinking body have been shaped in their relations with the world. Consider again the finger and hand movement of the expert typist, who no longer needs to look at the keyboard to re/produce a text on the computer monitor. These movements have been shaped by the geometrical configuration of the (QWERTY or other) keyboard such that the hands type without requiring the guidance from the perception of the keys. Those same hands, however, do many other, quite different things equally well, such as handling knife and fork while eating, holding a pen while writing, or tying shoelaces. Thus, the competence related to typewriting – here standing for any cultural practice and thinking – are not found in the structures of the body, hand or the brain, but in the observable composition of its actions in real, objective activity with and among other bodies (e.g. hands and keyboard). Thought is not found in the structure of the brain. Instead, to find it one has to investigate 'the "inorganic body of man", the "anatomy and physiology" of the world of his culture, the world of the "things" that he produces and reproduces in his activity'.[33] The example of the QWERTY keyboard – which is not a 'natural form' but one created by humans – also shows where the Spinozist position falls short: in failing to recognize that human beings not only are subject to conditions, learning by contemplating the world, but actively change this world to give it new form. These new forms give rise to specifically human, that is, societal forms of consciousness, which is a central point that Karl Marx makes when, in *Das Kapital* (Capital), he develops his theory of (exchange-) value, the *supersensible* pole of the *sensible-supersensible* commodity entering an exchange relation.[34]

The upshot of these considerations is that thinking is not a property of the individual body but of nature as a whole. Moreover, in the Marxian take, the specifics that distinguish humans from other beings are their societal relations and society. Therefore, 'a body of smaller scale and less "structural complexity" will not think. Labour is the process of changing nature by the action of [societal] man, and is the "subject" to which thought belongs as "predicate"'.[35]

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[32] Wolff-Michael Roth, 'The Emergence of Signs in Hands-on Science', in *International Handbook of Semiotics,* edited by Peter Trifonas (Dordrecht: Springer, 2015), 1271–1289.
[33] Il'enkov, *Dialectical Logic*, 74.
[34] Karl Marx and Friedrich Engels, *Werke Band 23: Das Kapital: Kritik der politischen Ökonomie Erster Band Buch I: Der Produktionsprozeß des Kapitals* (Berlin: Dietz, 1962), 85–98.
[35] Il'enkov, *Dialectical Logic*, 74. The original uses the equivalent of 'societal man [obščestvennogo čeloveka]' rather than 'social man' that appears in the English translation. See Evald V. Il'enkov, *Di-*

The Spinozist take, reread through the Feuerbach-Marxian lens, also can be found in late 20th century practice theory, where it is recognized that 'the world is comprehensible, immediately endowed with [sense], because the body, which, thanks to its senses and its brain, has the capacity to be present to what is outside itself, in the world, and to be impressed and durably modified by it, has been protractedly (from the beginning) exposed to its regularities'.[36] It is precisely because we are in, surrounded by, and part of the world that we also comprehend it. That is, whereas 'the world encompasses me, comprehends me as a thing among things … I, as a thing for which there are things, comprehend this world. And I do so (must it be added?) *because* it encompasses and comprehends me'.[37] We acquire practical knowledge and control of the encompassing space because we are materially included in the space, even though we may not notice it expressly and maybe even repress it. As a result, the social and material structures of the world that we inhabit are 'sedimented' as forms of dispositional structures, that is, in the forms of expectations and anticipations characterizing our being in the world. That comprehension arises from the openness of the body to the world, where it may be impressed as any other thing. 'Having the (biological) property of being open to the world, and therefore exposed to the world, and so capable of being conditioned by the world, shaped by the material and cultural conditions of existence in which it is placed from the beginning, it is subject to a process of socialization of which individuation is itself the product, with the singularity of the "self" being fashioned in and by social relations'.[38] Important to this practice-theoretic understanding is the rereading Marx enacts of Spinoza. The latter only recognized the impressions that the world makes on the body, whereas Marx specifically states that consciousness arises in and from the active changes that the thinking body effects in the world. Unsurprisingly, therefore, Bourdieu refers to Marx's 'Theses on Feuerbach' and the distinction that is made there between all preceding materialisms, which recognized only a passive side of comprehension, and articulates the necessity to develop a theory that recognizes the active side by means of which a socialized body produces the very conditions of which it is itself the product. In the end, therefore, the practical sense is inhabited by the world that it inhabits. The practical sense allows humans to act appropriately, *comme il faut,* in the heat of the situation, without having to stop and reflect. The thinking body – having been shaped by and while shaping the world – has developed dispositions that allow it to engage in a world immediately endowed with sense, in correspondence with the field. The dispositions, structured by the active and passive experiences of the world, are structuring in the perceptions of and actions in and toward this world.

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*alekti*č*eskaja logika: o*č*erki istorii i theorii* [*Dialectical Logic: Essays on its History and Theory*] (Moscow: Izdatels'stvo politi Izdatels'stvoeskoj literatury, 1984), 54.

[36] Pierre Bourdieu, *Méditations pascaliennes* (Paris: Seuil, 1997), 163.

[37] Bourdieu, *Méditations*, 157.

[38] Bourdieu, *Méditations*, 161.

From Vygotsky's Readings of Spinoza

Vygotsky programmatically wrote that he would bring Spinoza to life in Marxist psychology. This was so because the philosopher provided a foundation for a *monist* psychology that was to replace both dualistic and parallelist takes on the mind–body relation. Vygotsky noted that the problem of other approaches in bringing together mind and body ended up creating some form of theory of unity and failed to establish it as theory of identity of the psychological and physical. He found the greatest idea of the philosopher, one that 'liberated all of psychology', in the acknowledgment of the relative power of the soul (Lat. *animus*, soul, nowadays is rendered as 'psyche' or 'mind'). That is, 'the relationship between the soul and the body (<u>life</u> and <u>intellect</u>) are not absolute and immutable, but changeable, relative'.[39] The opportunities that arise from such statements for psychology are studies of the 'changes in the proportionate role of the soul in the life of the body, and of the intellect in life'; and, most importantly, 'everything in Spinoza breaks with mechanical causality, with immobility (there is no development, with disconnectedness, with parallelism and <u>demands</u> an escape beyond their boundaries'.[40]

Vygotsky apparently did not tolerate *inter*action, for parallelism, though inherently false, still constituted a consistent theory. Underlying the historical consideration of the teaching of emotions is the reading that – as different but parallel substances – soul and body could not act upon each other (i.e. *inter*act), because the two manifestations would be external to each other. Vygotsky followed the idea of the underlying unity, which would be articulated and developed in the notion of the thinking body by Il'enkov and Merab Mamardašvili, two materialist dialectic philosophers (see chapter 2). In fact, it would be to Il'enkov to provide a well-grounded philosophical justification of numerous key tenets of Vygotsky's cultural-historical theory and developmental teaching | learning (obučenie).[41]

Spinoza notes that '*the human mind does not perceive any external body as actually existing except through the ideas of affections of its own body*'.[42] We find this same idea, which makes the connection between the material world and thought, in the personal, fragmentary notes of Vygotsky. Thus, following the articulation of thinking and speech as the core problem of psychology, he notes, 'only through this issue can one correlate thinking and brain-functioning, just as only through the brain and its motion (which is what embodies psychic momentum) can one correlate thinking and the law of conservation of energy, thinking and a change in the direction of a moving point without expenditures of energy, thinking and catalyzing processes'.[43] Gestalt psychology is critiqued because it bases itself

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[39] Lev S. Vygotsky, in Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive: New Findings – Notebooks, Notes, and Scientific Journals of L.S. Vygotsky (1912–1934)', *Journal of Russian and East European Psychology* 48(1), 39.

[40] Vygotsky, 'Family Archive', 39.

[41] Vasily V. Davydov, 'Vklad E. V. Ilyenkova v teoretičeskouju psixologiju [E. V. Ilyenkov's contribution to theoretical psychology', *Voprosy Psixologii* (1994: 1): 131–135.

[42] Spinoza, 'Ethics', 261.

[43] Vygotsky, 'Two Fragments', 94.

on two different series of concepts, whereby the structure of the brain is set equal to the structure of inner experience. This, Vygotsky notes, does not bring psychology closer to a materialist understanding of the psyche. Speech, having both material (phonetic) and ideal (semantic) dimensions, is the key to the psychophysical (mind–body) problem. Indeed, being sensible–supersensible in the same way as commodity, and serving human exchanges in the same way, we can think of words as Marx was conceiving of commodity.[44] This then allows us, as Vygotsky notes, to overcome the body–mind dichotomy. In his notes, he relates 'the entire psychophysical problem' to a solution grounded in the reading of *The German Ideology*: 'Cf. Marx: the curse of matter on pure consciousness is moving layers of air, i.e., intercourse with the aid of language, rather than a connection with the brain!'.[45] He emphatically adds the German equivalent of 'very important'.

For Vygotsky, 'the psychophysical problem', 'the question of the spiritual and material in human consciousness', 'consists in the relation between thinking and speech', a problem that manifests itself in 'a parallelistic correlation of previously severed attributes'.[46] Vygotsky is interested in the *real* unity rather than the imaginary one that is generated by means of a correlation of externally related things. The guiding idea is situated in 'Marx: the materiality of consciousness in its link to language', which 'is historical materialism (its concrete principle) in psychology', so that 'thinking and speech are the central problem and the *via regia* of all historical psychology'.[47] Moreover, related to the Spinozist topic of will, he notes that it 'must be derived from … soc. relations internalized and embodied in the activity of the centers [cortex, subcortex] with use of natur., organic subordination (sublated category, actuating mechanism'.[48]

Vygotsky was particularly interested in a Spinozist take on emotions because the philosopher had articulated an idea of the person from a 'peak' standpoint. This idea – which 'is the true idea, because it is in agreement with its object'[49] – would be the guiding one for his Marxist psychology. We observe the previously articulated Spinozist take on the correspondence between categories and natural phenomena. Vygotsky died shortly after having written these notes; and he therefore never could articulate a theory of emotions, leaving only a preparatory text 'The Teaching about Emotions'.[50] One of his students, Alexei N. Leont'ev, achieved such a categorical reconstruction of the psyche, whereby acting, sensing, and emoting came to be irreducible manifestations of life beginning with the simplest of organisms.[51]

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[44] Wolff-Michael Roth, 'A Dialectical Materialist Reading of the Sign', *Semiotica* 160 (2006): 141–171.

[45] Vygotsky, 'Two Fragments', 95.

[46] Vygotsky, 'Two Fragments', 93–94.

[47] Vygotsky, 'Two Fragments', 94. *Via regia* is Latin for 'royal road'.

[48] Lev S. Vygotsky, in Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive (1912–1934): New Findings', *Journal of Russian and East European Psychology* 48(1), 29.

[49] Vygotsky, 'Family Archive', 41.

[50] Lev. S. Vygotsky, 'The Teaching about Emotions: Historical-Psychological Studies', in *Collected Works Vol. 6* (New York: Plenum, 1999), 71–235.

[51] Alexei N. Leontyev, *Problems of the Development of the Mind* (Moscow: Progress Publishers, 1981).

An important aspect of the relationship between thinking and speech is the recognition that speech is not the externalized result of a finished thought but that thought becomes itself in speaking. Speaking has an important bodily aspect that makes it different from the semantic, ideal aspects (i.e. "meanings") that are attributed to words. As speech unfolds, thinking develops so that the mind grasps its thinking only when speaking is finished. This is actually an old idea, which we may find in Spinoza's Proposition II.23, which states: '*the mind does not know itself except insofar as it perceives ideas of affections of the body*'.[52] Here we need to keep in mind that the affections are the modes themselves. For there to be ideas, affections of the body have to exist, and these ideas of the affections allow mind to know itself. Hence Vygotsky's articulation: thought exists once affection, material speech, has come to conclusion.

An important feature of Spinozist method exists in the way Vygotsky relates 'word meaning', 'thinking', and 'speech'. For Spinoza, the order in two modes of a substance is the same not because these somehow interact but because they are fundamentally united in *substance*. Because the whole is present in all of its parts, therefore, word meaning, if indeed it is a whole, has to be present in its modes (parts), thinking *and* speech. But this whole is different from the parts, which only one-sidedly manifest the whole and, therefore, are lacking in this or that aspect – precisely those that exceed the sum of the parts.

Vygotsky and the Feuerbach-Marx Connection

'It is not fortuitous that Spinoza's profound idea only first found true appreciation by the dialectical materialists Marx and Engels'.[53]

Karl Marx and Friedrich Engels took as their starting point for understanding human behavior Spinoza's recognition of thought and extension as two attributes of the same substance – as evidenced in the response that the latter gave to the philosopher Georgi Plekhanov, 'old Spinoza was quite right'.[54] Spinoza already emphasized *thought* as an attribute of substance rather than of the individual mind – which is only a finite mode. For Marx, thought, consciousness, is a *societal* product from the beginning, and remains such as long as humans exist. This idea was initially articulated in the works of the Spinozist philosopher Ludwig Feuerbach. These influenced Marx in his early years, such as when we wrote that '*society* is the complete unity of man with nature – the true resurrection of nature – the accomplished naturalism of man and the accomplished humanism of nature'.[55] The

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[52] Spinoza, 'Ethics', 260.

[53] Il'enkov, *Dialectical Logic*, 43.

[54] Georgi V. Plekhanov, *Selected Philosophical Works Vol. 2* (Moscow: Progress Publishers, 1974), 339.

[55] Karl Marx and Friedrich Engels, 'Ökonomisch-philosophische Manuskripte (1844)', in *Werke Band 40* (Berlin: Dietz, 1968), 538.

society-producing relations *among* humans determine the relations between humans and nature, and the relations of humans to nature determine the society-producing relations among each other. A Marxian psychologist later would note, accordingly, that we do not impose human significations upon nature but we disclose nature from these significations.[56] In this statement, we observe the Aristotelian conception of the apophantic nature of the statement, which is to let something show itself to be seen. Abstracted from their connections within the ensemble of relations of society, which also give shape to the ways in which humans relate to nature, human beings would think as little as a brain disconnected from the body of the organism. Thought, the ideal, therefore is not an individual psychological fact but instead a specifically human image of objective reality in human activity, which exists in the form of collective, societal consciousness. The life of the human ancestors, who became human during anthropogenesis, did not start with language in the way we know it today. They used sounds in the way animals use it, as a manifestation and function of activity, such as when the cry is used to spread affect. Marx and Engels exhibit the irremediable connection between the emergence and development of language and the development of human society. This point was not lost on Vygotsky, who notes that in individual development, 'the word did not exist in the beginning. In the beginning was the deed. The formation of the word occurs near the end rather than the beginning of development. The word is the end that crowns the deed'.[57] He thereby captured that children from the beginning of their lives – and even before, while still being in the womb – are exposed to the sonorities of language and then participate in a world structured, among others, by the sound-words that accompany and constitute activity. It is out of being with others in the word that consciousness arises. This precisely is the Marxist take, for 'consciousness [*Bewußtsein*] never can be anything else but conscious being [*bewußtes Sein*]'.[58] Consciousness always follows life, emerges from life, but does not determine life.

Vygotsky was deeply influenced by his reading of *The German Ideology*, which was published in Moscow, first in German (1932) than in Russian (1933), that is, he accessed this work during the final period of his life. This can be seen in his use of direct quotations from and paraphrases of this work – which have been omitted and are unacknowledged in the English translation of *Thinking and Speech* – and in parts of the text that have been excluded in the version most Anglo-Saxon readers know. These excluded quotation marks and text have been included in the following quotation, which constitutes part of his articulation of those aspects that *Thinking and Speech* has not covered but lie at 'the threshold of a problem that is broader, more profound, and still more extraordinary than the problem of thinking'.[59]

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[56] Georges Politzer, 'Les fondements de la psychologie: psychologie mythologique et psychologie scientifique', *La Revue de la Psychologie Concrète* 1 (1929), 27.

[57] Vygotsky, 'Thinking and Speech', 285.

[58] Marx and Engels, *Werke Band 3*, 26.

[59] Vygotsky, 'Thinking and Speech', 285.

'The consciousness of sensation and thinking are characterized by different modes of reflecting reality. They are different types of consciousness. Therefore, thinking and speech are the key to understanding the nature of human consciousness. But if "language is as ancient as consciousness itself", if "language is consciousness that exists in practice for other people and therefore for myself", when "the curse of matter, the curse of moving layers of air hangs over consciousness from the beginning", then it is not only the development of thought but the development of consciousness as a whole that is connected with the development of the word. Studies consistently demonstrate that the word plays a central role not in the isolated functions but the whole of consciousness. In consciousness, the word is what − in Feuerbach's words − is absolutely impossible for one person but possible for two. The word is the most direct manifestation of the historical nature of human consciousness'.[60]

We note that the psychophysical problem is at the heart of the matter, the fact that language and consciousness are weighted down by the 'curse of matter', an aspect that is completely absent in the modern day concern for 'meaning' that is even less material than language in its various instantiations. The descendance of Vygotsky's position at the end of his life from the works of Marx and Engels is apparent in the following quotation concerning the materiality of human consciousness.

'But even from the outset this is not "pure" consciousness. The "mind" is from the outset afflicted with curse of being 'afflicted' with matter, which here makes its appearance in the form of agitated layers of air, sounds, in short, of language. Language is as old as consciousness, language *is* the practical, real consciousness that exists for other men as well, and only therefore does it also exist for me; and language, like consciousness, only arises from the need, the necessity, of intercourse with other men. … Consciousness is at first, of course, merely consciousness concerning the *immediate* sensuous environment and consciousness of the limited connection with other persons and things outside the individual who is becoming conscious of itself; it simultaneously is consciousness of nature, which first confronts men as a completely alien, all-powerful and unassailable force'.[61]

The important aspect of this take is that society and societal relations are a condition for the emergence of consciousness in the form of language, which is an integral part in the material life of society, the real process of labor that is oriented to the satisfaction of basic and extended (not hedonistic, but real, life-preserving) needs. For Marx and Engels, society is that unit that allows us to understand the individual; and, as stated in the Eighth Thesis on Feuerbach, 'all societal life is essentially *practical'*. What is separate in the theories that tend to mysticism, the

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[60] Lev S. Vygotskij, *Myšlenie i reč': psixologičeskie issledovanija* (Moscow: Gosudarstvennoe social'noèskonomičeskoe isdatel'stvo, 1934), 318.
[61] Marx and Engels, *Werke Band 3*, 30–31.

individual and the social, bodily and mind, exist in the true unity of human, socie-
tal praxis and in the grasp of this praxis. This grasp is the consciousness of practi-
cal life, which operates at two levels. On the one hand, consciousness of relations
with other people precisely constitutes the higher psychological functions that
Vygotsky eventually would emphasize; and researchers themselves, rather than
mystifying these functions, can comprehend the origins and nature of these func-
tions by investigating their appearance anthropologically in real, concrete praxis.
This allows comprehending sensuousness as practical activity.

In its foreword to *The German Ideology*, the Institute for Marxism-Leninism of
the Central Committee of the Communist Party of the Soviet Union describes the
nature of the text to be foundational for a number of social sciences conceived
from a Marxist point of view. Thus, the text is foundational because it relates the
creation and development of language with the material life of society, human la-
bor. The Institute for Marxism-Leninism notes that Marx and Engels emphasize
the irremediable *unity* of language and human thinking and that the truth and reali-
ty of thought manifests itself in language (speech): 'Marx and Engels explicate the
nature and function of thinking, the intellectual *needs, interests, inclinations and
feelings of man*, they show that the decisive causes for their change and develop-
ment are grounded in material life of society, and they thereby lay the foundation
for a Marxist, dialectical-materialist psychology'.[62] In his discussion of the prob-
lems of the classical social sciences, Vygotsky points out the separation of the in-
tellectual from the material aspects of language, intellect was abstracted from a
whole that also has affect as one of its attributes. His own writings reflect his read-
ings of these passages from Marx and Engels. In the classical approach, thinking
was theorized in itself, divorced from life, 'from the living *motives, interests, and
inclinations of the thinking person*'.[63] To understand a thinking *person*, however,
requires more than considering what happens in the mind. This is so because 'to
explain the event we call "thinking", to disclose its effective *cause*, it is necessary
to include it in the chain of events *within which it arises of necessity and not fortui-
tously*. The "beginnings" and the "ends" of this chain are clearly not located within
the thinking body at all, but far outside it'.[64]

The concrete human psychology that Vygotsky envisioned conducts determinis-
tic analyses of thinking, which presuppose 'the identification of its *motive force,
the needs and interests, incentives and tendencies* that direct the movement of
thought in one direction or another'. The motive force of thinking does not lie in
thinking itself. Thus, 'it is not thought that thinks: a person thinks'[65], an idea that
Vygotsky has taken from Feuerbach, who writes: 'Man thinks, not the "I", not rea-
son'. Philosophy and psychology therefore have to have as their objects the '*real
and whole being of the person*', for 'the *reality*, the *subject of reason* is man alone'
and '*only man is the true and the real*'.[66] It is in this totality of human existence

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[62] Marx and Engels, *Werke Band 3*, x, emphasis added.
[63] Vygotsky, 'Thinking and Speech', 50.
[64] Il'enkov, *Dialectical Logic*, 37.
[65] Vygotsky, 'Concrete Human Psychology', 65.
[66] Ludwig Feuerbach, *Sämtliche Werke, Zweiter Band* (Leipzig: Otto Wigand, 1846), 339.

that we can find the unity of thinking and being, a fact that can be grasped only when human beings are considered as the foundation, the subject of this unity. Thought is not severed from the whole person only when we considered thinking not as a *subject for itself* but predicate of a real person.

Vygotsky's theoretical move thereby becomes consistent with the Marx's Second Thesis on Feuerbach, which states that the question concerning the concrete truth and reality of human thinking is not a theoretical but a practical one. It is therefore 'ridiculous to look for specific centers of higher functions in the cortex … they must be explained … in external terms, on the basis of the fact that man controls the activity of his brain from without through stimuli'.[67] It was indeed not Marx (and Engels) who realized that the truth of thinking is a practical question, but something that before them the philosopher Feuerbach had already noted. The latter already suggested that mind, consciousness, and rationality are not fixed attributes, not properties inherent to humans but only exist in and through activity in which they have a concrete practical function. As a result, the question about the nature of being always is and remains a practical question; and, reflexively, our being participates in this question. Something thought theoretically or talked about does not have to really exist, does not have to be a real, material being – a unicorn as a living animal is not part of our material reality just because we can paint one or describe it in words. This is so because in everyday praxis, human beings have to prove the truth and power, the this-sidedness and worldliness of their thinking. But because the word (language) is common to us, the thought of unicorns becomes a shared reality.

Feuerbach, although recognizing the role of the collective in the constitution of knowledge, nevertheless theorizes the individual as the seat of its essence; he thinks the essence of a person as his/her own essence. As a result, 'consciousness is self-action, self-confirmation, self-love, – self-love not in the sense of the animalistic – pleasure in its own perfection. Consciousness is the characteristic mark of a perfect being'.[68] This is not unlike the fundamental constructivist (enactivist) thesis that human beings are the result of their own constructions, as end products of their own actions bearing the same characteristics as the actions. According to Feuerbach, the *nature* of human beings is determined by the essence of the species, which is the *absolute essence* of the individual. The Sixth Thesis on Feuerbach states however that 'human nature is not an abstractum inherent in the single individual. In its reality it is the ensemble of societal relations'.[69] In the fragmentary text entitled 'Concrete Human Psychology', Vygotsky paraphrases this dictum comes in this way: 'the *psychological* nature of man is the totality of [societal] relations shifted to the inner sphere and having become functions of the personality and forms of its structure'.[70]

Society is produced by, and produces, the ensemble of human relations. 'Where a relation exists', Marx and Engels write, 'it exists for me, the animal does not "*re-*

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[67] Vygotsky, 'Concrete Human Psychology', 59.

[68] Ludwig Feuerbach, *Das Wesen des Christenthums* (Leipzig: Otto Wigand, 1841), 9.

[69] Marx and Engels, *Werke Band 3,* 6.

[70] Vygotsky, 'Concrete Human Psychology', 59.

late" itself to anything, does not "*relate*" itself at all. For the animal its relation to others does not exist as a relation'.[71] To exist for me, the relation (re-) appears as such in consciousness. This idea is integral to the role of relations in Vygotsky's approach, and he therefore takes any higher psychological function to have been relations with others before; that is, before becoming a psychological function, a function first *was* a real, social relation between two concrete human beings. Vygotsky does not state that there was something *in* the relation that then was internalized. Instead, the function itself was at one point a relation between people. In this approach, speech is paradigmatic, for it is both a social relation and a psychological means. In the case of speech, the connection between social relation and psychological function is stated even more strongly: 'All forms of verbal communication between adult and child later become psychological functions'.[72] Thus, whereas the formulation 'any higher function was the social relation between people' does not imply that all social relations will be higher functions, this latter formulation describes development in the case of verbal communication. The important role of social relations leads Vygotsky to take up some ideas from another Marxian psychologist and philosopher working in France, Georges Politzer, who conceived of a *psychology in terms of drama*.

Vygotsky was impressed but never really took up the conceptualization of drama, though a recent commentator has made a strong case that this idea was indeed underlying the psychologist's approach.[73] Drama takes into account the whole life as it plays itself out in particular situation, for example, where there are relations between people. In a relation, one person responds to another so that the origin of her thinking does not lie in herself but comes from the outside, from what affects her. The end point of her action does not lie in her, but the action – something she does or says – also is on the outside, another person or thing. This is a fundamentally Spinozist conception. The same is the case for the other person, who responds to the person responding to him. Doing and talking takes into account the other, who has acted before and who will be the recipient. Any saying and doing – points to be developed later in this book – therefore begins and ends on the outside of the person in relation with another. This is why drama is neither internal nor external: it always has front and back stage, and neither can be understood without the other. Drama therefore constitutes the locus of development, and the contradictions within the unfolding drama are a manifestation of both movement and force within the dramatic event. Thinking the human psyche in terms of drama takes psychology away from the traditional foci: that of scientific psychology, which seeks its object in the *world of nature*, and that of interpretive psychology, which seeks its object in the *world of the mind*. Vygotsky's approach, therefore, is consistent with the analyses that appear in *The German Ideology*, which orients its readers to the whole person, whose life determines her consciousness. 'Once the object of psy-

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[71] Marx and Engels, *Werke Band 3*, 30.

[72] Vygotsky, 'Concrete Human Psychology', 58.

[73] Nikolai Veresov, 'Zone of proximal development (ZPD): The hidden dimension?' In *Språk som kultur – betydningar i tid och rum,* edited by Anna-Lena Østern and Ria Heilä-Ylikallio (Vasa: Åbo Akademi, 2004), 13–30.

chology is defined as drama, the totality of the individual becomes the initial and fundamental hypothesis, without which no fact and no notion of psychology is conceivable, and the elementary analysis becomes not only possible but also really fecund'.[74] As a result, concrete psychology tends toward its parts that are dramatic in themselves. That is, when concrete psychology decomposes the drama, it does so on the assumption of the totality of the individual cannot be reduced, in the same way as facts that reveal themselves in the decomposition. A dramatic conception of mathematics is developed in chapter 9.

On Method

There are considerable implications for doing research when we take Vygotsky's theoretical approach. In this section, some of these implications are developed around the notions of the primacy of praxis, unit analysis, and dialectics and the implications thereof for doing 'concrete human psychology'.

Primacy of Praxis

We may distinguish *praxis*, the actual, sensual labor through which things get done from practice, the *conception* of the customary ways in which humans do things as it appears in the minds of people. The distinction is the same that Marx makes in denouncing all preceding materialisms, which capture sensuality only in the form of the *object or intuition*; but no as *sensual human activity, praxis*; not subjective'.[75] Concrete human psychology is concerned with how real living people, in the here and now of their concrete lives, make, and are made by, the everyday world that they inhabit; concrete human psychology attempts to understand how people succeed in doing what they do without having a conception or representation thereof in their minds. Thus, if we see a red light, we stop at the intersection. We do not see the light, interpret it, and then direct the relevant organ to push or pull the brakes. It is this orientation that we find in the writings of the late Vygotsky, who directs us to 'identify the needs, interests, incentives and tendencies that direct the movement of thought' rather than divorcing thinking 'from the full vitality of life'.[76] To do so, we need to orient toward praxis as the starting point of our research. That is, there ought to be a primacy of praxis when we attempt to understand mathematical thinking, learning, and development.

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[74] Politzer, 'Les fondements', 59–60.
[75] Marx and Engels, *Werke Band 3*, 5.
[76] Vygotsky, 'Thinking and Speech', 50.

Above I note the Marxian realization that consciousness is conscious being. First there is once-occurrent, always novel and in part unexpected being[77]; and then consciousness thereof follows. As a consequence, 'it is not consciousness that determines life, but life that determines consciousness'.[78] Marx and Engels distinguish between the two ways of proceeding. The first position begins with the consciousness as the living individual, whereas with the second perspective, analyses begin with real practical life, real, sensuous living human beings and then consider consciousness only as the consciousness of these beings. That is, concrete psychology needs to investigate psychological facts in terms of the segments of a drama, each segment itself dramatic through and through; and each act always implies the individual considered *as a whole*. In this way, social scientists can stop speculating and begin doing positive science that begins with real, sensuous life. This science is concerned with representing what people practically do, which leads to practical processes of human development. Scientists can stop stating empty phrases about consciousness, replacing these by real knowledge. For example, rather than pronouncing themselves about knowledge and knowledge construction, scientists would investigate real participation in human activity to learn about the ways in which individuals make thinking available for each other in the public forum and the ways in which order is produced *for the purpose to be seen as such by others*. When people queue, they are not just placing themselves in an already existing structure based on some mental schema but they are in fact acting *for others to see* them as lining up rather than doing something else. When individuals do something as mathematicians, then they act in ways so that others see them do *mathematics* rather than something else. That is, *anything that we recognize as mathematical is observably displayed in a public arena; thus, mathematics is out there in the open and we do not have to get into the head*. It is only when we force students to work on their own that mathematics *appears to be individual*, when in fact it still is social given that it *was* a social relation first. Students in school, especially those experiencing difficulties, may have to be reminded that they have to exhibit the mathematics of their mathematics, for example, when in the exam instructions requesting them: 'show your work'.

For Vygotsky, it is apparent that psychology needs to follow the route that Marx and Engels have outlined for the procedure of history. Contrary to idealist history, which tends to search for categories in every period, a Marxian conception of history 'remains constantly on the real *ground* of history, does not explain praxis out of ideas, explains formation of ideas out of material praxis'.[79] Thus, contrary to idealist psychology, concrete psychology does not seek recourse to 'meanings' to explain social life and how individuals contribute to its production, but uses the unfolding drama of social life to explain whatever consciousness arises.

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[77] Mikhail M. Bakhtin, *Toward a Philosophy of the Act* (Austin: University of Texas Press, 1993); also, '<K filosofii postupka>', in *Sobranie socinenij t.1* (Moscow: Izdatel'stvo russkie slovari jazyki slavjanskoj kul'tury, 2003).
[78] Marx and Engels, *Werke Band 3*, 27.
[79] Marx and Engels, *Werke Band 3*, 38.

The fundamental problem of classical psychology in both of its versions, and, with it, in classical educational research, is that the psychological facts are derived from the transpositions of life into the world of the text. Such a transposition destroys the drama of the drama. Thus, interpretive psychology and interpretive research more generally take as their object the lexical words that a person produces and derive from it their conjectures about the contents, structures, and processes of the mind. But all of this work is concerned with objects that appear in the world of text writ large, which no longer is the dialogical and transactional world that we inhabit, a world in which there is no time out (to think), but a world in which actions are ascribed to individual actors and the constitutive social relations have been reduced to *self*-action or *inter*-action.[80] Thus, to do *concrete* psychology, we have to return to the drama of social life, follow events as they unfold and investigate what people make available to each other that is taken up, transformed, and returned as part of the response.

Both Vygotsky and Politzer consider *geisteswissenschaftliche* (i.e. interpretive) psychology to be as reductionist as scientific psychology, even though the former tends toward understanding life as drama. Its problem arises from the fact that interpretive psychology is concerned with idealist 'meanings', psychological facts that result from the *mind*. Politzer is critical of the reliance on 'meaning' in interpretive psychology, which leads it to a primacy of the mind. Unsurprisingly, he uses the term 'mythological psychology' to denote this approach to psychology, which itself is opposed to 'scientific psychology' that only studies physiological and other bodily reactions. Concrete psychology, for its part, is concerned with the organization of how *praktische Menschenkenntnis* [practical ability to judge character] operates. It is therefore not unlike ethnomethodology, concerned with the ways in which everyday people not only produce the orderliness of their lives but also do so in ways that the order and the making thereof are available to other participants, who, in these expressions, recognize their own orderly, order-producing ways. To understand human beings, research needs to focus on the real societal relations, the real life conditions that humans produce and that produce them.

Marx and Engels note that there is a significant instant when the division of labor becomes such that material and mental labor are separated. At this point, 'from this moment onwards consciousness *can* really flatter itself that it is something other than consciousness of existing praxis, that it *really* represents something without representing something real'.[81] Before that point, the production of ideas, thoughts, representations, consciousness, and the likes is enmeshed with practical material activity and the material exchanges of real people – that is, it is enmeshed with the language of real life.

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[80] Ricœur shows how there are three forms of mimesis operating: the first, when we make and observe an order while participating in social life; the second, when social life is transposed into the world of text; and the third, when the world of text is transposed back into social life as part of the implications and applications of research results. Paul Ricœur, *Time and Narrative I* (Evanston, IL: Northwestern University Press, 1984).

[81] Marx and Engels, *Werke Band 3*, 31.

Theory and method are intimately related; they have to be related or there will be contradictions in the empirical results. Thus, if theory conceives of categories that are the smallest unit of thought that make sense, then research methods need to be devised such that they maintain the integrity of the analytic units chosen. If, for example societal activity is chosen as the minimal unit – as in modern, third-generation articulations of cultural-historical activity theory – then we cannot investigate mathematical behavior while being oblivious to schooling as the relevant unit that retains all the characteristics of society. We must not consider mathematical thinking apart from the fact that whatever a child or student says in a think-aloud or clinical interview session is part of an event organized for the specific purpose of collecting data concerning mathematical thinking. Similarly, if our research is to investigate moving phenomena, such as learning or development, then the method needs to take into account of the movement and not articulate a structural theory where entities remain stable and are changed only from without. Vygotsky, as Spinozist-Marxian thinkers generally, is interested in a dynamic account, which, inherently, has to be plural (multiplicious), for the phenomenon is changing *within* the unit chosen and, therefore, is non-self-identical. Finally, a theory that acknowledges societal relations as the essence of what it is to be human requires data and forms of analysis that do not reduce behavior to the individual but shows where anything apparently individual first has originated in a collective behavior. The following subsections elaborate on each of these constraints and affordances of method.

Unit Analysis

'Those things that are common to all things and are equally in the part as in the whole can be conceived only adequately'.[82]

Vygotsky charges traditional psychology with something that the science has taken over from philosophy: the separation of thinking from being. Thinking, rather than being a mode of acting in the world, thereby came to be transformed into an autonomous stream of thoughts. As evidenced in much of (social) constructivist theorizing, thinking itself is the thinker of thoughts rather than being in the service of a human individual while acting in the world. In his 'The Historical Sense of the Crisis', Vygotsky quotes from Feuerbach's text against the mind–body dualism, according to which 'the difference between *thinking* and *being* is not sublated in psychology. Even in regard to thinking one has to distinguish between the thinking of thinking and the thinking in itself'.[83]

The problem also exists when investigations start on the other end, with the body, when psychologists, as good natural scientist, focus on the ready-made things in the way that these appear to them in their present existence. 'Only then

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[82] Spinoza, 'Ethics', 265.
[83] Feuerbach, *Werke, Zweiter Band*, 353.

can the question arise: "How can spiritual, ideal thought emerge directly from the spatial interaction of soulless material bodies?"'.[84] The problem is eliminated as soon as there is a universal substance, Nature, which constitutes an integrality, which is the general foundation of the Extension–Thought unity. In the Spinozist take, theorizing does not begin with the parts (elements) but with the whole because the aim is 'to understand the place and role of every part, "mode", in the general plan of the whole, to understand the par as a separate *manifestation* of universal, integral Nature. The whole, in Spinoza's philosophy, is more than the sum of its parts, because it is present wholly in each one of its parts and creates the parts that are lacking'.[85] The late Vygotsky notes in one of the last texts he wrote before his death, the introduction to *Thinking and Speech*: 'a psychology that decomposes verbal thinking into its elements in an attempt to explain its characteristics will search in vain for the unity that is characteristic of the whole. These characteristics are inherent in the phenomenon only as a whole. When the whole is analyzed into its elements, these characteristics evaporate' because 'the internal relationships of the unified whole are replaced with external mechanical relationships between two heterogeneous processes'.[86] Thus, a psychology that begins with the elements inherently cannot get to the 'more than the parts' other than by introducing *external* forms that produce the relations between the parts. In the case of body and mind, Descartes introduced the pineal gland; Hegel trickily constructed a process of *sublation*; and modern-day psychology and mathematics education introduce tools and signs to function as the mediators between the two. But, Vygotsky charges, though united in the sign, the parts are theorized to coexist, but completely isolated one from the other. The same external mediators are required to bring together two externally related subjectivities in the pursuit of a common reality, intersubjectivity (see chapter 5). Vygotsky is interested in the 'concrete characteristics of the whole', which alone can explain the function of thinking and speech. Thus, an analysis needs to begin with a unified whole, such as the intersubjectivity in a social situation, to understand its parts, such as any individual or intrasubjectivity that arises and is expressed in that situation.

   The problem of method is nowhere better characterized for Vygotsky than when researchers 'smash' the unity of the physical and semantic parts of speech, that is, when research splits sound (sensible) from signification (supersensible). Readers clearly will be able to identify the massive amount of studies that focuses on signification ('meaning') while disregarding sound. When students' or teachers' bodies enter the analyses, then mediators, such as schemas, are required to make a connection between the movement of a hand and arm with the words that are taken to express 'their' 'meanings'. In this way, however, only an abstract unity is achieved, literally manifested in the abstraction that a schema represents. But, as Vygotsky notes, 'divorced from thought, sound loses all the unique features that are characteristic of it as the sound of human speech, the characteristics that distin-

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[84] Felix T. Mikhailov, *The Riddle of the Self* (Moscow: Progress Publishers, 1980), 66.
[85] Mikhailov, *Riddle*, 67.
[86] Vygotsky, 'Thinking and Speech', 45.

guish it from the other types of sound that exist in nature. … As a consequence, this research has not been able to explain why sound possessing certain physical and [psychological] characteristics is present in human speech or how it functions as a component of speech'.[87]

Vygotsky proposes to study human behavior in terms of 'units', which are the results of analysis that retain all the properties of the whole. Years before he articulates this aspect of his method in *Thinking and Speech*, he already encountered it in the Politzer text as presented above. Vygotsky thus proposes one such unit, the inner aspect of the word, its 'signification [značenie]' often translated as 'meaning'. The problem with the English 'meaning' is that it concerns the semantic aspect of the word alone. But such a merging must be possible in other languages as well, for Vygotsky points out that in most research, only the external, semantic side is studied. 'Word meaning', however, pertains to the *inner* aspect of the sound-word, which includes all features of the whole situation. As a result of focusing on external aspects, 'word meaning has been dissolved in the sea of all representations in our consciousness or our thinking'. Inherently, 'meaning', conceived as suprasensible in nature, therefore cannot be the unifying category. It cannot be, speaking with Spinoza, an adequate concept unless it is enlarged to encompass the semantic aspects as one of its modes. Thus, Vygotsky himself points out that značenie ['meaning'] is in excess of the semiotic aspect so that 'the <u>variation of meaning is</u> a deeper, more <u>essential, more internal analysis of a semiotic operation</u>'.[88]

In the Spinozist take characterizing Vygotsky's thoughts near the end of his life, that whole has to be such that it has sound and semantic aspects as its modes. A perhaps more fruitful way of proceeding is to focus on the function or value of the sound-word in the situation as a whole, an approach consistent with the Russian 'značenie', which also translates as 'value'. In the notes closer to his death, after reading *The German Ideology* and the prominence of consciousness, Vygotsky comes to orient toward *sense*. Thus, in 'documents from 1932, Vygotsky points out that the child is oriented toward sense virtually from the very first days of her life; her first questions are about sense rather than the meanings of her surroundings'.[89] He is interested in the complex dynamics of sense and how it manifests itself in thinking and speech. That sense has both visible and invisible aspects. Without the visible aspects, movement is incomprehensible. But there is more to sense because, in the case of movements, they exceed what is necessary and, therefore, are never precise. As a result, 'movement always has a latent, inner sense of movement, which always expresses <u>the person's attitude to the goal</u>, internal struggle, hesitation, additional goal, latent tendency or motivation, hot temper, weakness, exaggeration of the goal, attainment of the goal for show'.[90] Any movement may there-

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[87] Vygotsky, 'Thinking and Speech', 46.

[88] Lev S. Vygotsky, in Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive: New Findings – Notebooks, Notes, and Scientific Journals of L.S. Vygotsky (1912–1934)', *Journal of Russian and East European Psychology* 48(1), 55.

[89] Zavershneva, 'Notebooks', 44.

[90] Lev S. Vygotsky as quoted in Zavershneva, 'Notebooks', 53.

fore be in excess of (more than) what is required by the situation; but it also may be less than what is necessary. Vygotsky sees in this 'more or less' the key to the latent, invisible part of sense.

Dialectics

Vygotsky pursues an agenda of *dialectical psychology*, which 'proceeds first of all from the unity of mental and physiological processes'. He refers to Spinoza when stating that in dialectical psychology, mind is not situated outside nature, as it appears in Descartes' work, but is 'a part of nature itself, directly linked to the functions of the higher organized matter of our brain'.[91]

In his analysis of the historical sense of the crisis in psychology, a method-related and methodological investigation, the earlier Vygotsky relates many of the problems of psychology to method. He considers dialectics the most general, therefore most universal science, because it covers the universe in the way we know it, including nature, culture, history, and thinking. But he suggests that the principles of dialectics must not be imposed on the phenomena of interest from the outside. Instead, 'we must reveal the *essence* of the given area of phenomena, the laws of their change, their qualitative and quantitative characteristics, their causality, we must create categories and concepts appropriate to it'.[92] That is, Vygotsky argues for a data-based investigation of the phenomena and to show, through analysis, any dialectical principles at work. This is consistent with, and arises from the dialectical method, which insists on explaining the world from the world itself – a data-driven science that develops theory rather than a science that begins with theory and imposes it on the world – while leaving, if science is not yet advanced enough, the justification to the future. Thus, categories and concepts have to be appropriate to the phenomena, developed out of rather than foisted upon them. He alludes to the text of Engel's text on dialectics, which states that the idealists commit a serious mistake, which 'lies in the fact that these laws are foisted on nature and history as laws of thought, and not deduced from them'.[93] The result of this mistake is a world ordered according to categories and concepts, theory over praxis, rather than a set of categories and concepts that stand for ideas that are adequate to the natural phenomena in the Spinozist-Marxian sense. A second mistake coincides with the first, according to which the investigators do not realize the historical nature of the categories and concepts that they use.

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[91] Lev S. Vygotsky, 'Mind, Consciousness, the Unconscious', in *Lev S. Vygotsky Collected Works: Volume 3 – Problems of the Theory and History of Psychology* (New York: Springer, 1997), 112.

[92] Lev S. Vygotsky, 'The Historical Meaning of the Crisis in Psychology: A Methodological Investigation', in *The Collected Works of L. S. Vygotsky, Volume 3, Problems of the Theory and History of Psychology* (New York: Springer, 1997), 330.

[93] Friedrich Engels, 'Dialektik', in *Werke Band 20* by Karl Marx and Friedrich Engels (Berlin: Dietz, 1975), 348.

Engels articulates three main laws of dialectics, which are important for understanding how the earlier Vygotsky proceeds and what he articulates about psychological development: (a) the law of the transformation of quantity into quality and vice versa; (b) the law of the interpenetration of opposites; and (c) the law of the negation of the negation. Although Engels, following Hegel, uses natural (material) phenomena to exemplify this law, he suggests that it is equally valid for the history of human society and for living bodies, though it operates under complex conditions so that the quantitative relations are much more difficult to exhibit. These laws are highly appropriate and particularly useful in the context of Vygotsky's earlier agenda to address the psychophysical problem, that is, to overcome the apparent abyss between mind and body. This so because the second law concerns the interpenetration of opposites, here mind and body, which is achieved precisely by means of the Spinozist move to the one substance that has both modes as its attributes. The negation of mind by body, and of body by mind, thereby is negated. The first law would be of particular importance for describing developmental phenomena, where qualitatively new functions – Vygotsky calls these *neoformations* – arise from quantitative changes in old functions.[94] As a consequence of the later Vygotsky, the second law needs to be rethought in terms of a plural singular, where the emphasis is on plurality rather than dualistic oppositions; and the third law needs to be rethought starting with the plural singular, which inherently manifests itself as plurality.

We actually find the same kind of thinking already in Spinoza's texts. This philosopher is concerned with a flaw of a lot of research that exists to the present day, whereby concepts and methods are imposed on phenomena from the outside. Method is defined in advance of the research rather than being developed to be appropriate to the phenomena; and theoretical concepts are imposed on whatever researchers are looking at rather than developing methods and concepts (categories) through a thorough investigation of the phenomena themselves. Spinoza is critical of this approach when he notes that

> 'men are wont to form general ideas both of natural phenomena and of artifacts, and these ideas they regard as models, and they believe that Nature (which they consider does nothing without an end in view) looks to these ideas and holds them before herself as models. So when they see something occurring in Nature at variance with their preconceived ideal of the thing in question, they believe that Nature has then failed or blundered and has left that thing imperfect'.[95]

In attending closely to what people – students and teachers – actually do in mathematics lessons, we can find out not only the order of the world that they inhabit seen through their eyes but also, and more importantly, the methods they use

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[94] See Wolff-Michael Roth, '*Neoformation*: A Dialectical Approach to Developmental Change', *Mind, Culture and Activity* (2016); and Wolff-Michael Roth and Alfredo Jornet, *Understanding Educational Psychology: A Late Vygotskian, Spinozist Approach* (Dordrecht, The Netherlands: Springer, 2017).

[95] Spinoza, 'Ethics', 320–321.

for producing and exhibiting this order and its production. It is precisely in their mutual constitution within the system as a whole – which, in the dialectics of nature, is the Spinozist substance that is *causa sui* – that we understand social phenomena, because 'the transaction is the true *causa finalis* of things'.[96] Thus, it is likely that other social sciences are in the same boat as sociology, where, as Pierre Bourdieu suggests, researchers tend to chose as objects of research problems that involve the social order, discourses, and ways of thinking and ordering the world *without investigating the origin of the object, problems, and discourses*. Rather than imposing theoretical concepts and methods – whether these derive them from the everyday world or from the specific discourses of the scientific community – researchers need to be aware not only of the preconstructed concepts, which surround them everywhere, but also the tools of the construction. Investigative endeavors, therefore, first and foremost need to investigate the pre-constructed objects, that is, their own discourses, interests, and the historical processes of the construction of the former. If investigators do not do so, then the research risks '*borrow[ing] its problems, its concepts, and its instruments of knowledge from the social world*', and then records as data, facts, or representations that 'are the *product of a prior stage of science*. In short, it records itself without recognizing itself'.[97] The same task is posited by a cultural-historical psychology of the type that Vygotsky envisioned. The research reconstructs the present, which initially appears in thought in a superficial way, in terms of pre-constructed, generally disconnected determinations. It is at that point that we enact the most radical break from the formal methods that impose special order-producing (scientific) methods. Not surprisingly, therefore, Vygotsky tends to investigate the history of the problems that he chose, such as when reviewing 'The Teaching of Emotions', which was to become the preparation for his work on a Spinozist-Marxian take on affect.

   Researchers need to constantly attend 'to the details of the research procedure', which 'should have the effect of putting you on notice against the fetishism of concepts, and of "theory", born of the propensity to consider "theoretical" instruments … in themselves and for themselves, rather than to put them in motion and to make them *work*'.[98] A typical move in a cultural-historical approach – which Vygotsky held to be the 'maximal universal science' – is to return to the most general 'cell form' of the phenomenon and develop it together with the associated, adequate category. In Spinoza's words, this means that 'in examining natural phenomena we first of all try to discover those features that are most universal and common to the whole of Nature, to wit, motion-and-rest and the rules and laws governing them which Nature always observes and through which she constantly acts; and then we advance gradually from these to other less universal features'.[99] Given that Nature continuously evolves, this inherently requires a historical approach, which is rec-

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[96] Friedrich Engels, 'Notizen und Fragmente', in *Werke 20* by Karl Marx and Friedrich Engels (Berlin: Dietz, 1975), 499. *Causa sui* is 'cause of itself', whereas *causal finalis* translates as 'final cause'.

[97] Bourdieu, 'The Paris Workshop', 236.

[98] Bourdieu, 'The Paris Workshop', 228.

[99] Baruch Spinoza, 'Theological-Political Treatise', in *Complete Works* (Indianapolis: Hackett, 2002), 460.

ognized to be the chosen method to any scientific problem, among others because it allows us to understand how the situation of interest and associated discourses came to be in the first place. It is in this light that we can make sense of Vygotsky's endeavor to identify the genetic roots of the problem he investigates, whether it was the relationship between thinking and speech or the problem of the emotions. This would allow him to begin with the internal unity of a phenomenon that has thinking and speech as two of its modes. He thus begins with positing a complex whole, a 'dynamic meaningful system that constitutes *a <u>unity</u> of affective and intellectual processes*'.[100]

On Methods for a 'Concrete' 'Human' Psychology of Mathematics

With respect to materialist philosophy, Marx, all the while taking on its achievements, also critiques it as being inconsistent with their content. 'Even in the case of philosophers', he writes, 'who give systematic form to their work, Spinoza f[or] i[nstance], the true inner structure of the system is quite unlike the form in which it was consciously presented by him'.[101] Thus, 'the chief defect of all previous materialisms (including Feuerbach's) is that the object, reality, sensibility, is conceived only in the form of the *object or of intuition* but not as *sensual human activity, praxis,* not subjectively'.[102] The result of this is that the active aspect of activity is developed by idealist philosophers, who do not conceive of the real, sensual labor of people but instead approach it as the result of dealing with the objects in thought and then acting on them in the world. The kind of psychology Vygotsky envisions is a *concrete human psychology*, where the 'concrete' and 'human' are to be taken literally, in the sense in which we encounter everyday life. It is for this reason that he focuses on sociogenesis and the articulation of higher psychological functions that first appear *as* collective behavior and then become individual, no less social behavior. Experimentally, then, this individual behavior can be unfolded into collective behavior – not unlike what in more recent years has been realized by means of breaching experiments or the study of the social in 'breakdown' situations. Fundamentally, the invisible labor underlying the order and rationality of everyday human behavior becomes visible in problematic situations where what we normally do and assume no longer works. Members to any setting then exhibit the orderly and order-producing work for each other so that they can subsequently return to the normal way of going about life. This sort of research no longer deals with representations, mental frameworks, or meanings but with the real, sensuous labor of people, who are subject and subjected to the order that they themselves produce visibly for others as much as for themselves.

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[100] Vygotsky, 'Thinking and Speech', 50.

[101] Karl Marx, 'Marx an Ferdinand Lassalle – 31. Mai 1858', in *Werke Band 29*, by Karl Marx, Friedrich Engels (Berlin: Dietz, 1978), 561.

[102] Marx and Engels, *Werke Band 3*, 5.

To understand thinking, we need to study its functional relation to the activity as a whole. The relevant social and material order does not just exist out there but is continuously produced and made visible in the ongoing transactional work that also produces the object/motive of the societal activity currently being realized. In situation, people conduct this work together and for each other. Thinking is an integral part of this work, and its study needs to be oriented toward understanding its relations and functions. The methods do not need to be different than those that the people themselves employ for producing and exposing the relevant order of things. Microanalyses, therefore, are not conducted for their own sake, as distinct from the analysis of macro-level social structures. Instead, microanalyses are directed toward the structure of the joint work that produces the visibility of the social. The people doing such work in our studies are understood as the particular staff that produces the social phenomenon of interest. In this approach, there no longer is a distinction between method and the system, for research reveals the structure of the work that produces the phenomena that other research presupposes as a fact.

We may investigate human in a concrete manner behavior generally and mathematical cognition specifically by focusing on whole persons, inherently involved in an aspect that is part of the fullness of their lives. Vygotsky finds appealing the idea that we experience life as drama and in dramatic ways. Rather than investigating thought and thinking through the study of 'conceptions', 'mental frameworks', 'beliefs', 'meanings', 'interests', 'motivations', and the likes we are better off investigating how people, in flesh and blood, conduct everyday life such that it can be seen as reasoned and rational. For this, we actually have to pursue what people really in life do rather than what they are made to do and reason in events transposed to the level of the text. To elaborate this distinction, it helps to invoke the concept of *mimesis* that allows distinguishing between different forms of text.[103]

Everyday human behavior is patterned even when the 'representations' and verbal 'meanings' are absent, as can be seen in the ways we move and do certain things. But we also talk; and this talk has as its first and foremost function to let any order be seen and to orient toward its orderly production. That language, therefore, is part of the everyday world but also (sometimes) makes it present again, that is, it is a copy of the world. We may denote this form of imitation of actual practice that occurs in praxis as *mimesis$_1$*. In most research, this order and its achievement is not of interest, but it is precisely the point of ethnomethodology, which constitutes a radical alternate to all other forms of research, qualitative and quantitative research combined. This radical nature of the alternate can be understood as soon as it is recognized that interpretive and scientific psychology (of mathematics education) do not investigate and operate on the real worldly phenomena but on a transposition thereof that exists in narrative and representational forms. Whatever the method for producing and identifying the order in terms of scientific concepts, they are different from what ordinary people do. This is why scientists have to state the methods they use; and this is why the methods sections take such an important

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[103] The following exposition of mimesis$_1$, mimesis$_2$, and mimesis$_3$ is derived from chapter 3 of Paul Ricœur, *Time ad Narrative Vol. 1* (Evanston: Northwestern University Press, 1984), 52–87.

position in quantitative as much as in qualitative scientific research articles. On the other hand, if the methods of identifying and producing order are not different from what we do in the everyday world then there is no need to specify them.

The order in the narratives – which may be in terms of 'meanings', 'positioning', or 'discourse' of qualitative studies or the reaction times, variables and measures, and latent constructs so dear to quantitative studies – then is a second form of mimesis referred to as *mimesis$_2$*, the mimesis characteristic of a created world, the world of the text. We then see that mimesis$_2$ captures the world of theory, where things work on paper, constituting abstractions from and of the real world that we inhabit and the order of which is made available as part of our lives in and as mimesis$_1$. This gap was precisely a key issue in the work of Marx, 'who avoided introducing as an explanatory principle abstractions which had no empirical functions, and which could not be vindicated by observing the ways in which human beings actually behaved'.[104]

Finally, mimesis$_3$ refers to the way in which our everyday world comes to be reconfigured by the narrative productions in the textual world of mimesis$_2$ (theory). This is where we may locate the complaints of practitioners, which are concerned with the 'irrelevance' of theory to practice. Readers will be familiar with such complaints from teachers, who often state that what they learn in university courses is of little use in their actual praxis; and what their praxis is little if at all captured in the theories that they encounter in their university courses. The situation is similar in the case of other fields, such as when apprentice electricians encounter one kind of mathematics in their college courses – e.g. the trigonometry required for calculating the geometry of electrical conduits with paper, pencil, and calculator – and on the job site, where they use tools engraved with markers and numbers that allow them to bend real conduit in much less error-prone ways.[105]

In this approach, mimesis$_2$, 'the configurating operation constitutive of emplotment is a result of its intermediary position between the two operations I am calling mimesis$_1$ and mimesis$_3$, which constitutes the two sides [*l'amont et l'aval*] of mimesis$_2$'.[106] In the original, Ricœur uses the terms 'l'amont', upstream, and 'l'aval', downstream, which better than 'the two sides' that appears in the translation gives the sense of the nature and relation of the three ways in which – and types of situations where – order is produced. The distinction between the three loci of order production allows us to understand our own research endeavors. In my own praxis (analysis), manifested throughout the chapters that follow, I exhibit the practical order production and practical exhibition of order and ordering work in the world of the participants – i.e., I am concerned with mimesis$_1$ – rather than drawing on special methods to translate the participant world into a world of text to create mimesis$_2$ and then to analyze the structures within mimesis$_2$.

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[104] Sidney Hook, *From Hegel to Marx: Studies in the Intellectual Development of Karl Marx* (New York: Columbia University Press, 1994), 277.

[105] Wolff-Michael Roth, 'Rules of Bending, Bending the Rules: The Geometry of Conduit Bending in College and the Workplace', *Educational Studies in Mathematics* 86 (2014): 177–192.

[106] Ricœur, *Time ad Narrative*, 53.

Because we experience life as drama, Politzer and Vygotsky suggest developing psychology in terms of drama. If any higher psychological function is a relation with another person *first*, then the object of the psychologist no longer is an abstraction but the real relation of two people. This relation, as any human relation, can be expressed in dramatic form and in terms of recognizable roles – e.g. parent–child, child–teacher, child–peer, etc. Rather than establishing mythologies, psychology would then become entirely 'concrete', dealing in and with the kinds of things, phenomena, and concepts that are from everyday life; and it would be completely 'human', for drama is *essentially* human. It may not surprise that for Aristotle drama (tragedy and comedy) played such an important role, for this literary form takes its central concerns from real life, depicts them in real life forms, and returns them to real life in recognizable form when audiences appreciate recognizing in and learning from represented life, their real life, which they now better understand and for which they learn in turn.

2

The Thinking Body

'Spatiality may be the projection of the extension of the psychological apparatus. No other derivation probable. Instead of Kant's a prior conditions of our psychological apparatus. Psyche is extended, does not know thereof'.[1]

'*Essentia mentis in hoc consistit … quod sui corporis actualem existentiam affirmat* [The essence of the mind consists in this … that it affirms the actual existence of its body]'.[2]

In radical constructivism, the mind (culture) is reduced to the physical body (biology). Thus, as Ernst von Glasersfeld points out, '[Humberto] Maturana set out to describe and explain all the phenomena that are called "cognitive" from a biological foundation'.[3] Similarly, the German radical constructivist philosopher and biologist Gerhard Roth, in his country often discussed together with Maturana, reduces the mind ('Geist') and consciousness to physical states of the body generally and to the brain and its neuronal circuitry specifically.[4] However, Vygotsky already realized that saying something like 'consciousness is a function of the brain' actually 'points to the theory of parallelism', whereas the focus on physiology of the brain constitutes a materialist (biological) foundation.[5] Spinozist-Marxian philosophers completely reject the reduction of consciousness to the brain, while

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[1] Sigmund Freud, *Gesammelte Werke XVII* (Frankfurt/M: Fischer, 1999), 152.

[2] Benedicti de Spinoza, *Ethica: Ordine Geometrico Demonstrata et in Quinque Partes Distincta in Quibus Agitur*, chapter 3, retrieved from http://users.telenet.be/rwmeijer/spinoza/ works.htm; Baruch Spinoza, 'Ethics', in *Complete Works,* trans. Samuel Shirley (Indianapolis: Hackett Publishing, 2002), 284.

[3] Ernst von Glasersfeld, 'Die Unterscheidung des Beobachters. Versuch einer Auslegung', in *Zur Biologie der Kognition* ed. Volker Riegas and Christian Vetter (Frankfurt: Suhrkamp, 1990), 284.

[4] See, e.g., Gerhard Roth, 'Enstehen und Funktion von Bewußtsein', *Deutsches Ärzteblatt* 96 (30) (1999), A-1957–1961.

[5] Lev S. Vygotsky, 'The Historical Meaning of the Crisis in Psychology: A Methodological Investigation', in *The Collected Works of L. S. Vygotsky, Volume 3, Problems of the Theory and History of Psychology* (New York: Springer, 1997), 290.

showing that such a reduction never could sufficiently take into account the double nature of human relations to one another and the material world.[6] Parallelism, the idea that there are two different lines of foundation and development of mind and body, is a form of dualism. In contrast, Sigmund Freud, the author of the introductory quotation, aphoristically noted on his deathbed that the psyche is extended and that there is no other explanation. Most importantly perhaps, psyche does not know about its extension. Here, then, at the end of his productive scholarly life, Freud comes to a conclusion in which there no longer is a chasm between the mind – the modern version of Aristotle's ψυχή [psyche] and the Roman's *anima* – and body (extension). Psyche is extended body, and the extended body is thinking. In this chapter, I present the notion of the *thinking body* in the way it has arisen from a Marxian reading of Spinoza in the way that Vygotsky had envisioned it.

Historical Philosophical Context

The currently reigning, classical approaches to theorize mathematical thinking focus either on the individual mind (sometimes constructing for itself what was constructed in the group) or on the individual body (enactivism, embodiment). In both type of approaches, the result of mathematical activity are structures that then determine what the individual does. This includes concepts (conceptions), schemas, mental structures, and the likes. In the Spinozist approach, when read through a Marxian lens, mathematical actions are understood in terms of the thinking body, which does not require 'any ready-made schemes of actions within it' to perform knowledgeably in the world.[7] Moreover, thought is understood in a *systemic* manner, that is, as a {thinking body | nature} manifold. This point is developed using classical examples from Spinoza to Il'enkov and a 21st century mathematics classroom.

To make an argument against subjective psychology Vygotsky quotes Feuerbach: 'what for me [or *subjectively*] is a spiritual, immaterial, supersensual act is in itself [or *objectively*] material, sensuous'.[8] This is a central point of Marx's analysis of commodity, which has both a sensuous, practical aspect as apparent in its use-value, and supersensible aspect, as apparent in its exchange-value (or simply *value*). In a Marxian approach, these two aspects are but manifestations of the double nature of sensible–supersensible nature of commodity.[9]

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[6] See, for example, Karl Marx and Friedrich Engels, *Werke Band 23: Das Kapital: Kritik der politischen Ökonomie Erster Band Buch I: Der Produktionsprozeß des Kapitals* (Berlin: Dietz, 1962), 85–98; Evald V. Il'enkov, *Dialectical Logic: Essays on its History and Theory* (Moscow: Progress Publishers, 1977); and Merab Mamardašvili, 'Analysis of Consciousness in the Works of Marx', *Studies in Soviet Thought* 32 (1986), 101–120.

[7] Il'enkov, *Dialectical Logic*, 51.

[8] See Vygotsky, 'Crisis', 324. He omitted the words in square brackets. See Ludwig Feuerbach, *Sämtliche Werke, Zweiter Band* (Leipzig: Otto Wigand, 1846), 350.

[9] Mamardašvili, 'Analysis of Consciousness', 105.

The keystone to the Spinozist-Marxian system, and, pertinent to the present context, to our understanding body and thought in mathematics, is that

> 'There are not two different and originally contrary objects of investigation – body and thought – but only *one single* object, which is the *thinking body* of living, real man … only considered from two different and even opposing aspects or points of view. Living, real thinking man, the sole thinking body with which we are acquainted, does not consist of two Cartesian halves – "thought lacking a body" and a 'body lacking thought'. In relation to real man both the one and the other are equally fallacious abstractions, and one cannot in the end model a real thinking man from two equally fallacious abstractions'.[10]

Thought and extension are two attributes (properties) of the same substance (Nature). By means of the movement of thought, 'Spinoza cut the Gordian knot of the "psychophysical problem", the mystic insolubility of which still torments the mass of theoreticians and schools of philosophy, physiology of the higher nervous system, and other related sciences'.[11] It is 'in man, in the form of man, in his person [that] *Nature itself* thinks … Nature thinks *of itself*, becomes aware of *itself*, senses *itself*, acts on *itself*'.[12] Acting materially upon the world also is a mode of action of the body, this time expressed spatially. Because thinking and acting materially are two modes of action of the same substance, there cannot therefore be the kind of cause-and-effect relationship that is typical of much of psychological and educational research, whereby a person is said to think and then act upon what has been thought. This is so because thinking and bodily acting are different modes of action of the same body that manifest (express) themselves in different form and, therefore, are considered in the light of different aspects (attributes). Just as we can see only a duck or a rabbit in the duck–rabbit drawing (Fig. 1.1), our considerations tend to be oriented to thinking or doing.

What then is the relationship between my material body and my thought? From the Spinozist-Marxian perspective taken here, the relation is that of an organ to the mode of its action. A body thinks in the way that eyes see, legs walk, and ears hear. As a result, the thinking body does not cause changes in thought, it cannot cause changes in thought because it cannot act *on* thought: the very existence of the thinking body as thinking *is* thought. The analogy with other organs and their modes of acting immediately makes this description intelligible: a thinking body does not cause changes in thought, as the eye does not cause changes in perceiving, as legs do not cause changes in walking. If the thinking body acts, it does not do so because some thought preceded it: the activity of the thinking body is thought in the same way as the activity of the eyes is seeing. As a result, thinking is not the outcome of an action, is not something produced, for it is the action of the thinking body itself considered under the attribute of thought. Thinking does not cause spa-

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[10] Il'enkov, *Dialectical Logic*, 31.
[11] Il'enkov, *Dialectical Logic*, 33.
[12] Il'enkov, *Dialectical Logic*, 34.

tially expressed changes in a body, such as when a mathematician moves a hand along a straight-line graph signifying a linear function between two variables, but exists and manifests itself through that movement. In the same way, changes brought about in the body by its relation to other bodies constitute a change in its mode of activity and, therefore, in its thinking. Thus, 'all talk of thought first arising and then "being embodied in words", in "terms" and "statements", and later in actions, in deeds and their results, all such talk, therefore, from Spinoza's point of view, is simply senseless'.[13]

From this follows that thought does not and cannot cause its articulation in speech, because thought is the end product of a process of thinking that develops itself in the course of speaking (saying). Once *Saying* has ended, thought is available as *Said*. Vygotsky, critiquing psychology for taking language to be a mirror of thought – a critique that he had encountered years before in a text by Georges Politzer – writes that thought is completed rather than expressed in the word and 'man actively participates in his relations with the environment and through the environment he himself changes his behavior, subjecting it to his control'.[14] The thinking of the thought expressed does not cause thinking in the same way that the seen does not cause the process of seeing the seen. About the relationship between thinking and its control, Vygotsky notes that humans control their activity through the use of tools and signs, which stimulate the brain in certain ways. In this move, there is a double relation: one between thinking and body, for the stimuli are of material nature that affect the brain and another between thinking and thinking, for in activity, as a mode of action of the body, the use of signs is associated with the change in another mode of action of the body, thinking. As a result, I do not construct thoughts: they occur to me, which I experience as my thinking.

The definition of a thing in general and a mathematical thing in particular must explain its essence; and we must not replace such a definition with some property or attribute. Spinoza, conveniently for the present purposes, uses a mathematical object, the circle, as an example for the definition of a created thing. If the circle were to be defined as a figure in which all lines connecting the center and the circumference are equal, we would have described a property rather than the essence of the circle. The essence of a thing includes its proximate cause and, when considered in itself, it contains all properties. In the case of the circle, this leads us to two requirements. First, it 'would have to be defined as follows: a figure described by any line of which one end is fixed and the other movable'[15], a definition that clearly encompasses the proximate cause. Second, from the preceding definition, we derive the property of the equality of the lengths of all lines from the fixed point to the resulting drawn figure. Here, then, the circle has been defined in terms of the real, practical, and sensuous actions of a real human being. In that situation, thinking is the ideal part of the overarching, real, sensuous activity, which also has

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[13] Il'enkov, *Dialectical Logic*, 44.

[14] Lev S. Vygotsky, *The Collected Works of L. S. Vygotsky. Vol. 4: The History of the Development of Higher Mental Functions* (New York: Springer, 1997), 59.

[15] Benedict Spinoza, *Complete Works* (Indianapolis: Hackett, 2002), 26.

a material part; and in this sensuous labor, both external nature and the person are changed.

The seventh proposition of Part II of the *Ethics* states a connection between material reality and categories that is central to the Spinozist-Marxian approach to categories: '*The order and connection of ideas is the same as the order and connection of things*' and provides the circle as an example: 'a circle existing in Nature and the idea of the existing circle … are one and the same thing, explicated through different attributes'.[16] As soon as the idea of the circle is given in the form of the preceding definition, a figure made by moving one end of a string held at a fixed point on its other end, the coincidence of the material thing and the idea is apparent. As a result, the same order will be found whether Nature is considered under the attribute of Extension or under the attribute of Thought. In his exposition of the Spinozist system, Il'enkov comments on this example: 'When I describe a circle with my hand on a piece of paper (in real space), my body, according to Spinoza, comes into a state fully identical with the form of the circle outside my body, into a state of real *action* in the form of a circle. My body (my hand) really describes the circle, and the awareness of this state (i.e. of the form of my own action in the form of the thing) is also the idea, which is moreover adequate'.[17] The difference between a compass, which makes a circle, and the human being lies, in part, in the mechanical aspect of the former and the enormous flexibility of the latter. Thus, Spinoza considers the human mind capable of perceiving many things*,* a capacity that is a function of the different configurations that the body can take. He proves this conjecture by means of two postulates that refer to the large number of affections of the human body and of ways of moving and disposing of external bodies.

As stated above, Marx differs from Spinoza and the philosophers who have taken up his monism, including Feuerbach. Where Marx differs considerably from Spinoza is this. Humans are not just acted upon by nature, passively being shaped in their relation with the natural world. Instead, humans change the world, also making tools to bring about further change. Thus, sensations are not merely the way in which humans experience things as these act upon the body, but they are a dimension of the transactional relations between an active, thinking body and its surroundings. The circle, which is never found in nature in the almost perfect form in which humans may produce it, is one example of a thing that humans created; and the mathematical idea of the circle was born in the endeavor to make the object increasingly perfect.[18]

In his commentary on Spinoza's example, Il'enkov points out the importance of the definition of a circle as a mode of construction the real thing so it becomes objective part of the world. Here, then, the discourse arises together with a set of real actions corresponding to it, and these real actions are those of a thinking body along the contours of the circle defined in terms of the discourse. The thing there-

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[16] Spinoza, 'Ethics', 247.

[17] Il'enkov, *Dialectical Logic*, 69.

[18] Edmund Husserl, 'Die Frage nach dem Ursprung der Geometrie als intentional-historisches Problem', *Revue Internationale de Philosophie* 1 (1939): 203–225.

fore exists for the individual not just in terms of sound-words (and even less in *idealistic* 'meanings'), but also in terms of the movements of its thinking body, that is, in terms of a kinesthetic 'melody' associated with the production of a real, material circle. As a result, 'the ideal exists there where there is a capacity to recreate the object in space, relying on the word, on language, in combination with a need for the object, plus material provision of the act of creation'.[19] Thought is an active capacity of production rather than the result of contemplation, and it exceeds passive reception, which, as shown in chapter 1 in the case of seeing a line, is not so passive at all.

The enigma lies in the fact that the light falling on the eye is experienced by a thinking body but not by a non-thinking body, such as a stone, even though it is equally impacted by the photons of light from that circle in Nature. Equally important is the fact that the circle is experienced as something outside of the body rather than something created within and by it. It is experienced outside not only by me but by all other human beings as well. It is not a construction of our mind but a real thing into which we may bump or that we may follow around the circumference with the index finger.
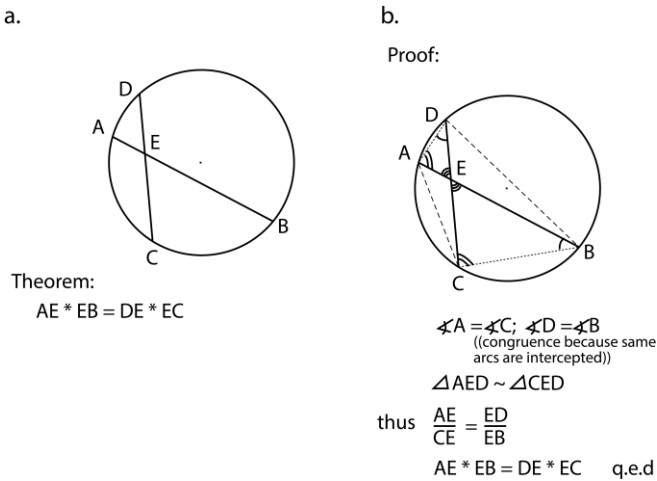
This approach to the thinking body and its objects of action has considerable epistemological consequences, for insofar as the circle is a natural thing, it must be explicated through the attribute of Extension and insofar as the circle is a thing in thought, it must be explicated through the attribute of Thought. A created thing, therefore, because idea and proximate cause of the material fall together, is the result of something that has both Thought and Extension as its attributes. There is only *one* substance, for two substances are by definition external to each other and therefore do not communicate. This one substance, living Nature, manifests itself in my *thinking body*. This body is not a piece of meat (material) in which the brain produces thought. Instead, the *thinking body* is in excess of my material body such that its one-sided manifestations are body and thought. For the purpose of further exemplifying and elaborating the notion of the thinking body, consider the following case relating to a theorem from geometry.

The relationship between thought and extension is apparent in the example of the theorem concerning any pair of intersecting cords of a circle. In any such case, the rectangles formed from the segments of each cord are equal (Fig. 2.1). These pairs of equal rectangles exist insofar as (a) the circle as such exists and (b) the idea of the circle exists. If we let one such pair exist, then the idea thereof exists 'not only insofar as they are merely comprehended in the idea of the circle but also insofar as they involve the existence of those rectangles', where Spinoza refers to the angle at E, which in his case was a rectangle.[20] The relationship also manifests itself in the relationship between a proof account, as it appears in Fig. 2.1b, and the doing of the practical labor of proving. Thus, once someone follows the steps indicated in the proof and obtains the same result, she 'knows' the relevant mathematics not merely theoretically (e.g. as something memorized) but indeed practically.

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[19] Il'enkov, *Dialectical Logic*, 264.
[20] Spinoza, 'Ethics', 248.

Fig. 2.1 **a** Theorem of the equality of rectangles formed from the segments of intersecting cords. **b** *Proof account* for the theorem of the equality of. *Doing* the real work described in the account will reveal the adequacy of thought embodied in it

a.

D
A
E
·
B
C

Theorem:
AE * EB = DE * EC

b.

Proof:

D
A
E
B
C

∡A = ∡C; ∡D = ∡B
((congruence because same arcs are intercepted))

△AED ∼ △CED

thus $\frac{AE}{CE} = \frac{ED}{EB}$

AE * EB = DE * EC   q.e.d

She knowledgeably *does* mathematics rather than *talking about* mathematics. Errors, inadequate ideas, 'result from the incorrect application of words to things' such that when 'somebody says that the lines joining the center of a circle to its circumference are unequal, he surely understands by circle, at least at that time, something different from what mathematicians understand'.[21] In this take, therefore, there no longer is a difference between theory and practice, knowing mathematics and doing mathematics. As a result, *we know mathematics when we can do it*, and we only repeat empty words in those cases that mathematics educators traditionally refer to as knowing something but being unable to apply it. From the Spinozist-Marxian take, 'when people say … that someone possesses knowledge but us unable to "apply" this knowledge to reality, they are making an essentially quite absurd statement'; for, we have to ask, 'how can anyone know an object – and be unable to relate this knowledge (knowledge of the object) to the object!'[22]

The Thinking Body: A Case from Elementary Geometry

The curriculum – designed together by the regular teacher of that second-grade class and one of the researchers – was premised on the idea that bodily experiences are constitutive of geometrical knowledge.[23] Among the many tasks, there was one in which children were given mystery objects located inside cardboard boxes, which they could only feel with one hand by entering through a hole but that they

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[21] Spinoza, 'Ethics', 271.

[22] Evald V. Il'enkov, 'Knowledge and Thinking', *Journal of Russian and East European Psychology* 45 no. 4 (2007), 76.

[23] For an extensive description see Wolff-Michael Roth, *Geometry as Objective Science in Elementary Classrooms: Mathematics in the Flesh* (New York: Routledge, 2011).

could not see because of a baffle. Working in small groups, the children were asked to use plasticine for building models of the mystery object. That is, they were to shape the plasticine so that it resembled the mystery object without actually having seen it. The description pays particular attention to one of the three girls in one of the groups (Melissa).

Doing 'Defining a Cube'

Melissa, after feeling out the mystery object for a while, begins to shape what will eventually look like a cube. She announces, 'feel it eh? I have felt it's a cube, huge', where the descriptive adjective 'huge' is articulate louder than normal and extending over more than one second. Jane and Sylvie object, insisting that it is not a cube, they have not felt a cube when it had been their turn to reach into the shoe-box. Yet Melissa insists, and provides a reason: She has checked the sides in a way that she demonstrates by means of gestures: Grasping her cubical model with the left hand, she uses an unchanging caliper-like configuration of the right thumb and index finger and holds it against three different, orthogonal edges of the cube (Fig. 2.2).

She does not merely describe what she feels but also provides a description of what she has done. In so doing, she in fact offers a set of instructions for what one *has to do* to feel the mystery object in the way she does. This is apparent when she moves her hand and arm through specific configurations (Fig. 2.3). But deploying such movements *in order to* symbolize movements that have occurred before requires the capacity for and memory of these movements. These movements are the same as when she was feeling out the mystery object, though now they have symbolizing function whereas the original movements had epistemic and ergotic (i.e., sensing and work-oriented) function. It is evident that the body does not have to have schemas to reproduce its own movements. We walk quite well without (consciously) placing our feet; and we ride bicycles very well without telling our bodies what to do. In the same way, Melissa's fingers, hand, and arm move because they have the capacity to move in this way, and this capacity has developed in the

Fig. 2.2 Melissa explains why the object is a cube. Using her plasticine model, she holds the fingers in a caliper configuration to three mutually orthogonal edges while saying that she had 'checked the sides like that'

Fig. 2.3 Movements of Melissa's hand and arm that she said to have used inside the shoebox to feel out the mystery object and which have led to the feeling of a cube

course of a life of moving this way. In both instances represented (Fig. 2.2 and Fig. 2.3), therefore, the illustration of what she has done is produced with the same hand: the representing movement is in fact indistinguishable from a presenting movement. The cube exists not primarily in the form of some abstract (geometric) idea but in the form of successive movements that rotate the cube and associated perceptions. In and with these movements, the sameness of the different sides is *instructably* made visible in the here-and-now of this situation. 'Instructably' here means that any person witnessing her demonstration, following what Melissa says, could have the same experience with the object in the box. Those movements are necessary and sufficient 'to warrant the judgment "here is a cube"'; and this judgment requires the 'displacements to be located in an objective space', which they are in and as of the demonstration and instruction.[24]

Despite this exchange, and pursuing the effort to convince each other, the three continue to talk about the nature of the mystery object. Melissa persists, maintaining that there is a cube, where we must understand that she is not producing empty words but that whatever she names cube is defined by the descriptions/instructions she has provided (Fig. 2.2, 2.3). At one point, while Jane reaches into the box, Melissa offers Jane the cube and asks her to feel it simultaneously with feeling the mystery object. Jane turns the cube while apparently doing the same within the box; she turns the cube and again places her hand on top; and she does so for a third time (Fig. 2.4). Jane orients toward Melissa and asserts, 'it does not feel the same'.

Some time later, the girls still have not come to an agreement. The teacher, having come to their table, requests arriving at a conclusion so that their results can be discuss in a whole-class meeting. At this point, Jane proposes to Melissa the same procedure that she has used before but with her model instead of the cube as the reference object. Melissa states wanting to feel it out: 'to see if it's the same'. While Melissa's right hand apparently moves about in the box, the video clip shows Melissa's left hand touching the rectangular prism. Jane places her palm on

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[24] Maurice Merleau-Ponty, *Phénoménologie de la perception* (Paris: Gallimard, 1945), 235.

Fig. 2.4   Jane feels the top of the cube while her right hand is in the box apparently doing the same. She turns the cube and again touches the top; and then she does the same with the third side

Fig. 2.5   Melissa feels out the rectangular prism on one side, then rotates it and feels out another side, and again, and again

the rectangular prism in Melissa's hand, instructing her which part to feel while doing the same with the (mystery object) in the box. Melissa then feels the prism on one side, rotates and feels it on the other side; she rotates and feels it out three more times (Fig. 2.5). While doing so, she continuously gazes intently toward her left hand and the rectangular prism in it.

At the beginning of this lesson fragment, Melissa, as the other children, was literally in the dark about what her hand would find out and what she would be learning. Her right hand was groping about, without visual access, to find out the nature of its contents. She could not intend what she will have learned once everything would have been said and done. When she eventually does withdraw her hand and begins to work on the plasticine, the mystery object will have given itself to her. To know, Melissa has had to feel out the material, that is, move her fingers and palm over the object to see what feeling the movement yields and how the material gives itself. What gave itself initially was a «cube», where the chevrons are used to distinguish a thing from its name, 'cube'. She could know what she had in her hand only after the exploration, opening up a gap between doing exploring something unknown and knowing the thing that will have been explored. Eventually, even before trying to build her model, she already gives it a name: 'It's a cube'. That is, she already knew her way around the world and associated languages to associate the feel in/of her hand with something of a similar cubical form as other entities that populate her everyday world.

Some mathematics educators might be tempted to say that this case is consistent with the Sapir-Whorf hypothesis, whereby existing linguistic categories – here 'cube' – shape human perception. Others might argue that it is an example of existing theories driving the nature and content of observation. But these would be hasty conclusions, for right up to the instant that the mystery object gives itself in and to her hand, she has nothing to go by for using a linguistic or perceptual filter that

makes it a cube (unless cubes are the only thing she knew, in which case everything would have been a cube). Right up to the moment that the mystery object was giving itself as what it is – so that Melissa *could* in fact give it a name – there was not even *something* (some *thing*). This object as such, with geometrical properties that could be talked about, did not yet exist until it actually had emerged, here literally from the dark and in her hand. This invalidates the constructivist suggestion that Melissa 'interpreted' *what* she had in hand, for 'to interpret', a transitive verb, requires an object. But that object did not exist before something had given itself to the sensing hand. What Melissa 'sees' and expresses in the cube is 'a conditioned thought, it is born "at the occasion" of that which arrives in the body, it is "excited" to think through it'.[25] Melissa could not construct or interpret a thing until after she had found herself holding *this* rather than another object.

At this point, Melissa was convinced that what she had felt was consistent with a kind of cultural object known under the name of cube. She did not just believe in, but had evidence in the form of descriptive properties of, a thing that can be felt and that her telling instructed others to feel when reaching into the box. In her account, there were three kinds of sides, with edges of the same length. She exhibited those invisible but sensible properties by using her model, turning it about, and holding the thumb and index finger in a caliper configuration that suggested a measurement. Melissa was convinced of the mystery object's cubical form right up to the instant when, after having explored the object for an eighth time, her voice marked surprise – 'Oh, oh', together with a facial expression that culturally competent individuals *see* as her surprise – that a new and different kind of object had emerged from the sensuous movements of her hand.

When Melissa began following Jane's instruction, intentionally oriented toward a cube, she did not intend this new and different form of the object. But the mystery object gave itself in a new way; and Melissa expressed this new way in the reconfigured plasticine material, as a rectangular prism with a pair of large faces separated by two pairs of narrow sides. There was therefore a qualitative shift in the experience, where an intentional orientation toward a cube was overturned and overturned itself to become the intentional orientation toward a rectangular prism. This new experience arose from her tact, which was, without knowing itself as such, already pregnant with this new form. In fact, there was continuity in the materiality of the mystery object, in its active-passive constitution, and discontinuity of form. It is in and through her grasping feeling that Melissa discovered in her hands the very object others had wanted to tell her about. It was in and through her feel right before and at the moment of surprise that she came to have in hand the rectangular prism that allowed her to discover the relevance of her peers' instructions to see and move/feel. That is, these descriptions that Jane and Sylvia had provided made sense *after* the mystery object had given itself as a «rectangular prism».

Before the first («cube») and second gift («rectangular prism»), there was a lot of talk intended to convince Melissa that she was wrong in asserting the mystery

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[25] Maurice Merleau-Ponty, *L'œil et l'esprit* (Paris: Gallimard, 1964), 51.

object to have cubical shape. However, all the previous verbal descriptions and perceptions of the models others had built were wrecked by what Melissa had truly felt rather than imagined and, therefore, could not convince Melissa of any incorrectness of her own sensations on the basis of which she had built a cube as a model of the mystery object. Now, as she turned Jane's model over and over to feel it out with her left hand while doing the same with the mystery object in her right hand, a new shape suddenly was born from her hand. The relevance of others' descriptions and models could only emerge from her actions and sensations in a manner that was not anticipated. She could know *what* the others were talking about when she had this *what* in hand. But because initially she did not know it, she could not intentionally orient toward the «rectangular prism» until it was in her hand as a new sensation, which emerged in excess of what she had perceived and, at the time, could perceive. Inside and outside the box, Melissa moved the mystery and model objects, respectively; and their (non-) correspondence was born from the associate feel and feelings. The particular shapes («cube», «rectangular prism») were born from and in a texture of movements, which constituted the invisible ground of the figures (shapes). These first phenomena of perception were passive, 'because they are absolutely foreign to the activity of the will or the me'.[26] Even the specific form of the activity was foreign. To be able to be repeated (e.g., to visualize it), any movement had to have arisen from a self-affection, which in turn made it possible to recognize the repetition in the absence of intention. No *me* would be possible and recognize itself if it were not for the immediate, internal sensation or apperception of the living flesh [Fr. chair].

Between the first («cube») and second birth of form («rectangular prism»), Melissa was confronted by talk inviting her to see and interpret differently what she was feeling. She was encouraged to reconsider the nature of the object in the light of the experiences of others, expressed in the material forms of their models. She was invited to reconsider what she had felt in the explicit confrontation of her model with those that Sylvia and Jane had shaped. Although there were perceptible differences, and although Sylvia considered her model to be (slightly) different from the one Jane had formed, their teacher explicitly named (and therefore classified) these as the same. The teacher not only said to Sylvia that hers 'will be a rectangular prism *as well*', but she also visually grouped the models by collocating Sylvia's and Jane's separately from Melissa's. The teacher insisted that they come to an agreement as to whether the mystery object felt more like Sylvia's or Jane's (while holding up each of the two models) or more like Melissa's. The teacher emphasized that there was only *one* mystery object so that there could not be *different* models. She said, 'as a group you have to say, "We think it is …"'.

Despite the insistent calls on Melissa to change her position on the nature of the mystery object – necessitated by the task condition to come to an agreement about what the *one* thing in the box was – the one she sensed in her hand, resisted being felt as something else: as what Jane and the teacher named 'rectangular prism', as

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[26] Pierre Maine de Biran, *Œuvres inédites, tome II : Essai sur les fondements de la psychologie* (Paris: Dezobry et E. Magdedeleine, 1859), 28.

what Jane and Sylvia perceptually articulated in their plasticine models, as what Sylvia exhibited to see in her iconic hand gestures and in her specific instructions of how to go about feeling the mystery object.

It was not as if Melissa had not been trying. Substantially more frequent and longer than Sylvia and Jane, she had reached into the box to move the object (as in Fig. 2.3) and to feet it out. She had made apparent her efforts to follow the instructions Jane and Sylvia provided to guide her explorations. But right up until the end of the eighth time she had reached into the box, the mystery object first felt and then remained what she expressly named 'a cube'; and she shaping her plasticine mass into a cubiform shape she also felt in the shoebox. In this shaping, her right hand, the one that has moved about in the shoebox and felt out the mystery object, also came to move about, feel out, and give shape to the mass of plasticine outside of the box. Although she apparently changed her mind in the end, initially and for the longest amount of time, what would turn out to be a «rectangular prism» (in the right hand) initially did feel like the plasticine cube in the left hand. But it was not her mind that had changed the thing. The mystery object had revealed itself to Melissa's touch in a different way and without (and perhaps against) Melissa's will; and when it revealed itself differently than what it initially appeared to be, Melissa announced this novel phenomenalization with and as a surprise.

From the perspective of Melissa (and other children like her) there is not a lack but a plenitude of experience that exceeds intention. Our actions are pregnant with new forms that exceed the associated goals (intentions). Vygotsky also realized this when he writes in his personal notes: 'we never make precise and only necessary movements … we do more or less than what is necessary in terms of the situation, the key to the latent sense is in this <u>more</u> or <u>less</u>'.[27] It is from this invisible more that the newly felt object is born. We now need to understand how out of this 'more', a plenitude, a new plenitude arises in/as an apparent excess of intuition over intention. Even and especially in the second phenomenalization of the mystery object as a «rectangular prism» there was an excess of intuition over the intention that was initially oriented to perceiving a cubiform object. Once she had felt it for a first time, Melissa found *that* object again and without trouble. During the second qualitative shift of perception, a new object phenomenalized itself *even though* and *despite* the initial intentional orientation toward a cube. As soon as Melissa had felt the mystery object as «rectangular prism», a path was laid toward remembering it as such (see next subsection). How something felt to be a «cube» all of a sudden can be felt as a «rectangular prism» is itself a problem to be addressed by research.

When the hand explores a surface of an object, there are two aspects: one related to the voluntary movement of the hand, the other related to the resistance offered to the hand on the part of the object. The two aspects are simultaneous. The double aspect related to the hand movement requires the consent of two parts of

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[27] Lev S. Vygotsky, in Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive: New Findings – Notebooks, Notes, and Scientific Journals', *Journal of Russian and East European Psychology* 48 no. 1 (2010), 53.

the same body, one of which has to be voluntarily movable. If we were to discover forms in this manner – whether the resistance is experienced as proper or foreign of the body – the intelligent being would appear to be the basis of geometry constructed because of capacities within itself. More than 200 years ago, the French philosopher Pierre Maine de Biran suggested that when the feeling hand is stopped in its movement, the individual knows or feels immediately an obstacle on the outside of the me, a cause outside of its will. The resistance to the free movement of hand or eye, arising within and from the organ itself, is at the origin of the ability to remember, imagine, and, therefore, to intend. The organ follows the object, *entrained* in its movement by the object, and in this movement experiences resistance to itself in an auto-affection of the flesh. This movement and the associated experienced resistance (which diminishes with habitude), the kinesthesia, are at the origin of the re/cognition of the object.

It is this self-affection of the thinking (because living) body that allows us to understand the episode involving Melissa. It is not thinking that determined the sensing hand, and the sensing hand did not *cause* the thinking. If cognizing the «cube» had been a simple issue of passive reception, then Melissa should have been in a situation to recognize the resemblance of mystery object and those that her peers showed her. She should have been able to follow their instructions and perceive what they had perceived. Her mind, thought, would then have been the determining cause of the bodily sense. If this was not the case then it was that in and as result of her actions, the «rectangular prism» did not emerge from the auto-affection related to the exploring hand-arm movements. But once the hand and arm constituted the appropriate movement, then that new object was born in and from it. That movement and the object are but two manifestations of the same relation, whereby the shape of the object brings about the shape of the organ's movement felt, but where the original impulse for the movement itself arises from thought. That movement-launching impulse itself has its proximate cause in the presence of the object in the box.

Movement Gives Form and Memory

In the preceding empirical study, forms («cube», «rectangular solid») were born and subsequently were associated with names ('cube', 'rectangular solid'). Initially, the form did not exist but came into existence as the thinking bodies followed the form of the mystery object that also was cognized as outside and as common to the three girls. Nature initially is bottomless ground that constitutes the possibility of the transition from the strictly unseen to the perceived form, a transition that for Melissa gave rise to the «cube» and then the «rectangular prism». What makes the first and second crossing possible?

The experience of form may be said to have brought perception to life rather than the other way around.[28] It is the world explored that accords to the perception this crossing, and indeed shapes the perception-producing movements. In the episode, it is the mystery object itself that accords to Melissa – as to her peers – the crossing from the invisible to the perceivable forms, leading first to the emergence of the «cube» and then to that of the «rectangular prism». These forms, born from the unseen, are not and cannot be the results of intentional construction, as constructivist educators (and philosophers) want to have it. Rather, it is the object itself that gives rise to the form. But how does the object do this? It is, as Spinoza and modern philosophers suggest, by transmitting to our perception its own movement as the condition to perceptually perceive the object again and again. The mystery object itself transmits to Melissa's hand its own movement. This movement is the imprescriptible condition for the unforeseeable «cube» and «rectangular prism» to emerge from the unfelt into the something that can be felt again and again; and, via the shaping of the plasticine model, from the invisible and unseen into the visible. To feel the «cube» or «rectangular prism» is to receive the «cube», «rectangular prism», since when something appears, it is the result of that something giving itself to be seen. As suggested above, Vygotsky was attuned to this 'more' that comes with every movement. Because of its theoretical relevance, the previous quotation is situated again in its broader context. Thus, Vygotsky notes:

> 'Unit of psychomotor development: semantic movement rather than sense + motor mechanism. … The visible and invisible sense of movement: visible – movement toward a goal (mechanically necessary) – and this sense, without wh[ich] movement is incomprehensible; but we never make precise and only necessary movements, so movement always has a latent, inner sense of movement, wh[ich] always expresses the person's attitude to the goal, internal obstacle, struggle, hesitation, additional goal, latent tendency or motivation, hot temper, weakness, exaggeration of the goal, attainment of the goal for show, etc. We do more or less than what is necessary in terms of the situation, the key to the latent sense is in this more and less'.[29]

How are the tangible forms born so that the children could turn their attention to them? Taking the mystery object, the hand (eyes) explores it. What the hand of the thinking body remember, however, are not the external forms it feels, but its own movements – which is a Spinozist-Marxian reading of our case. No memory *of* movement in the form of schemas or 'internal representations' is required to bring about the movement *again*. Rather, it is in and as of these movements themselves that memory exists – kinetic dynamics and kinesthetic memory are two aspects of the same phenomenon. It is like the case when the eyes move in one way, the rabbit is seen; and when they move in another way, the duck is seen (Fig. 1.1). The next time we look at the sketch, we see one or the other figure not because we have

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[28] Jean-Luc Marion, *Étant donnée: Essaie d'une phénoménologie de la donation* (Paris: Presses Universitaires de France, 1996), 76.

[29] Lev S. Vygotsky, from his personal notebooks, as quoted in Ekaterina Iu. Zavershneva, 'Notebooks', 53.

memorized some schema or concept but because the eyes make the same move-
ments again. These eye movements *are the consequence* of the eyes crossing in
following the lines in the sketch not the consequence of an interpretation. This is
why the verbal instructions and descriptions that Melissa hears cannot bring about
the actions and perceptions in her inquiry. Thus, in the absence of the object, we
can relive an internal sense the traces that the movement has left. All movements
that the hand makes, all positions that it takes in moving over the solid, later can be
repeated in the absence of the object and as wished. More importantly, these
movements can point backward to the elementary perceptions and the primary
qualities originally felt in the hand. When the hand recognizes a form, it is similar
to recognizing an earlier experience during a 'flash back'. This pointing can func-
tion to remember or recall the ideas; effectuated by means of available signs, such
recall constitutes memory in the strict way of speaking. It is by recalling the act of
taking the mystery object in, traveling over it with, the thinking body's hands not
only remembers the shape but also the affective engagement associated with it: '*It
is the memory of a body that remembers each time the manner of taking the [ob-
ject], to move toward to take hold of it*'.[30] But only a thinking body remembers, not
a material body as such. The material body in movement and thinking are but two
manifestations of the same thinking body that exceeds the materiality of the bio-
logical body. It is only by means of the thinking body that the reductionist tenden-
cies of the two psychologies – one biological, one interpretive (cultural) – that du-
alism (parallelism) can be overcome.

   In the present situation, we see that the situation is not simple, for Melissa re-
peatedly took the hidden object into her hand, felt it, and turned it about to be in a
position where she could build a model of it. This model, too, she held in her
hands. In fact, when asked to explain why she thought the mystery object to be a
cube, she took her cubiform model into the hands and showed, by means of a series
of movements with constant caliper configuration of her right hand, that the differ-
ent edges of the object were of the same length (Fig. 2.3). She maintained her posi-
tion on numerous occasions during this lesson, despite repeated attempts on the
part of Sylvia and Jane to convince her otherwise, and despite the attempts on the
part of the videotaping graduate student and the teacher requesting explications.
That is, an abyss opened between kinesthetic memory of the past and present kin-
esthesis. This is particularly interesting in light of the fact that Melissa's right hand
and arm moved in the way she also told to have done within the box – and where
she had felt a cube. She not only said that it was a cube but she built one, so that
both the movement and the feel existed in the presence outside the box. The body
itself was doing again what it had been doing inside (Fig. 2.3). Such a 'doing
again' recalls an earlier experience, but it is the idea in the present tense rather than
one represented in a different modality. This, too, points us to a form of intention-
ality, for to be able to produce the hand-arm movements as part of the communica-
tive act, the habitual movements of the thinking body constitute the 'recalling'. It is
only when the body already knows to move in a particular way that there can be an

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[30] Michael Henry, *Incarnation: Une philosophie de la chair* (Paris: Éditions du Seuil, 2000), 206.

intention to do so in the movement we observe. My work in science education has shown that it is precisely in this way that ergotic, work-related movements become symbolic movements.[31] The first movements over the mystery object, which came forth without intention, were the very material from which intention emerged (e.g., the intentions observable in her gestures). The preceding case study shows, however, that there is an obstacle or resistance to this sort of memory. Here, there was a delay between the first and second form – i.e., the movement and sensation concerning the mystery object *inside* the box followed by movement and sensation of the model object *outside* the box – so that what turns out to be different is actually feeling the same.

Instructed and Instructable Movements

'Tact [is] the geometrical sense. It alone, in effect, can give a base to these originary synthetic observations of the geometer, which recompose an intelligible solid with the point, line, surface'.[32]

In the end, however, it is the result of the simultaneous feeling of the mystery object in Melissa's right hand and her own model in the left hand from which a new perception is born: the mystery object feels, and therefore is, like the model that Sylvia had built and given to Melissa for verification. The difference between the hidden mystery object and the cubiform model was not born from the movements separated in time. The movements felt the same when she first held the one (original) and then the other object (model) in her hand(s). The movements and sensation felt differently when she had the opportunity to touch, move, and feel the two objects simultaneously.

The goal of education is tradition, the handing-down of ways of knowing typical of a culture. But the handing-down of cultural ways of knowing does not occur in the same way as we trade objects in the marketplace, where these pass seemingly unaffected from one hand into another. Rather, it is in and through their actions with material forms that new generations come to relive, in the present tense, the sense-constituting acts that lie at the origin of specific forms of knowledge such as geometry; it is in and through their actions that new generations bring to life geometry, and, simultaneously, keep geometry alive. The lessons from which the empirical materials derive were explicitly designed to provide children with experiences of living through such sense-constituting acts. Because geometry is grounded fundamentally in our bodies

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[31] Wolff-Michael Roth, and Daniel Lawless, 'Signs, Deixis, and the Emergence of Scientific Explanations', *Semiotica* 138 (2002): 95–130; Wolff-Michael Roth, 'The Emergence of Signs in Hands-on Science, in *International Handbook of Semiotics* ed. Peter Trifonas (Dordrecht: Springer, 2015), 1271–1289.

[32] Maine de Biran, *Œuvres inédites*, 145–146.

(cf., introductory quotation to this section), in experiences such as those we observe in the episodes, the children are doing rather than preparing for geometry by studying (texts about) geometry and geometrical objects. The task of modeling a mystery object exhibited an interesting phenomenon all-too-often glossed over in education and educational research: The relation between the different perceptual modalities (gaze, tact).

   The episode featured here foregrounds the relation between the visible and the tangible. The mystery object inside the box is invisible to the three girls, and they are required to build something in its image without actually having seen it. On the basis of tact, they initially have different experiences, which materialize in three different models. These models ground the different experiences in a way that is visually accessible to all (rather than somehow inaccessibly hidden in the minds and as interpretations). There is therefore an interlacing of the visible and the tangible. Using as his analytic phenomenon the hand that touches the other hand in the course of exploring the world, the philosopher Maurice Merleau-Ponty notes that one's own body is a thing of the same kind as the material world. The movements of hand, as the movements of the eye, are tied to the sensible and visible that they produce. He concludes that 'because … all experience of the visible was given to me in the context of the movements of the gaze, the visible spectacle belongs to the touch no more and no less than the "tactile qualities"'.[33] We therefore have to get used to the idea that the visible is carved into the tangible. However, whereas both the tactile and visible maps are complete, they are not confounded; they are complete but non-superposable parts. The episode shows the complex relations between the visible and tangible maps, and event the relation between the tangible maps of two objects.

   Whereas the teachers of the class wanted Melissa to shape her plasticine into a rectangular prism (i.e. they wanted Melissa to learn geometry rather than disadvantage her by leaving her with the 'incorrect' model), the issue is more complicated from the perspective of mathematics. Thus, for example, whether Melissa's model and the hidden mystery object – or the three objects Melissa, Sylvia, and Jane produced – are part of the same class is a matter of context. This is so because different contexts can be specified whereby the cube and the rectangular prisms are the same and others can be specified where even Jane's and Sylvia's models are different. For example, the cube is a special case of the rectangular prism, so that Melissa had indeed constructed a model thereof. Sylvia's and Jane's models are different when the proportions are considered; and both would have been inappropriate when the task had been to build scale models of the mystery object. All objects considered here further would have been members of the same class if topological relations had been the organizing feature – most mystery objects under consideration in the lesson, including cubes, (triangular, rectangular, hexagonal) prisms, spheres, pyramids, and (solid) cylinders, would have been in one class

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[33] Maurice Merleau-Ponty, *Le visible et l'invisible* (Paris: Gallimard, 1964), 174–175.

(homotopically speaking, these have zero holes), whereas cups, mugs, rings, tubes, and doughnuts would have been part of another (homotopically speaking, these have one hole).

On Construction, Embodiment, and Enaction

In mathematics education, there are many theories of knowing and associated theories of learning. One prominent theory is that of *radical constructivism*. But it cannot be an adequate theory because it reduces all behavior to the processes in the individual mind, which is informationally closed to the surrounding world. This approach constitutes the antithesis of what is described here, for it deepens rather than overcomes the psychophysical problem that Vygotsky aimed at overcoming, especially through an increased attention to the works of Spinoza after realizing that his own work was overly mentalistic. Several approaches that are supposed to deal with the mind–body problem have been taken up from the literature. These include *enactivist* and *embodiment* theories. There already exists a critique, which aims precisely at the specter of Cartesianism within that literature from the perspective of the phenomenology of movement. Thus, the term embodiment 'is little more than a lexical band-aid covering a three-hundred year old Western wound. In using the term, we are actually perpetuating a divide that has not healed and will never heal so long as the terms of the division remain part of our thinking'.[34] This may be attributed to the fact that we, humans, still do not know or have fathomed what it is and means to be the thinking bodies that we are. Sheets-Johnstone comes to the conclusion that the embodiment and enactivist literatures fail doing justice to movement itself and its animate forms. These literatures fail to recognize that movement has been there prior to anything that we consider to be a form of thought. Those movements, associated with self-sensing and sensing of the world constitutes the body of (word) sense. Similar issues arise in enactive approaches, which do in their discourses include important terms from dynamic, non-linear systems theory. But in these discourses, 'the language of an affective-kinetic experiential dynamics is, in contrast, nowhere to be found much less even noticed as missing in neurophenomenological and enactive approaches'.[35]

In what would be her first attempt, Melissa named, gesturally described, and modeled the mystery object (as) a cube. In a constructivist account, she might be said to incorrectly 'interpret' the object or to have a misconception about three-dimensional objects. Thus, (related to two-dimensional geometrical objects) children are said to go through a stage where they recognize the things they touch but

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[34] Maxine Sheets-Johnstone, *The Primacy of Movement* 2nd ed. (Amsterdam: John Benjamins, 2011), 310–311.
[35] Sheets-Johnstone, *Primacy*, 471.

are 'at first unable to abstract shape for want of sufficient exploration'.[36] This has been attributed to a 'want of sufficient' and 'inadequate exploration'. As a result of 'the lack of exploration' there is 'a general deficiency in perceptual activity itself', which is taken to mean 'that the child's perceptions are still passive or static instead of being integrated in a system of sensori-motor co-ordinations tending to bind them together'.[37] The same kind of language is used to describe the emergence of three-dimensional geometrical concepts. What is required, in this constructivist account, is a passage from one centration to another, a process of decentration. But as we can see in the movements of Melissa, this process is already present. We can see how the hand and arm movement is, if anything of the likes, from centration to centration, where the mystery object is felt in a specific way depending on its orientation. She is right at the «cube» itself; and the «cube» manifests itself in what she feels and sees. The mystery object initially is a «cube», denoted by the word ('cube') and the model. It is so not because of a *lack* of exploration – of which there were a considerable number over an extended period of time. But the very process of centration requires an object, and it, as material and form (shape), is given rather than intended. Indeed, Piaget's descriptions have been qualified as quasi-empiricist such that the 'passage to a higher type of perception and conduct could be simply explained by a more complete and more exact registering of experience'; he does so when development in fact 'supposes a reorganization of the perceptual field and the *advent* of clearly articulated forms'[38], which, here, are the «cube» and the «rectangular prism».

Piaget's position, though based on careful observational studies of children, therefore, does not help us understand learning and development or change in perception. He only articulates a lack rather than a developmental movement. Development is not the result of a lack being reduced, in an apparent teleological process whereby a void comes to be filled (i.e., the child developing *toward* adult rationality). Rather, what we need to understand is the fact that from the perspective of Melissa (and children like her) there is not a lack but a plenitude of experience that exceeds intention. Our actions are pregnant with new forms that exceed their intentions. We need to understand how out of any plenitude a new plenitude arises in/as an apparent excess of intuition over intention. Even and especially in the second phenomenalization of the mystery object as a «rectangular prism» there was an excess of intuition over the intention that was initially oriented to perceiving a cubiform object. Thus, Jane finds *that* object again once she has felt it for a first time.

As Piaget, modern-day constructivists often characterize children's knowing negatively: as lack, deficit (with respect to adult), or deviance (e.g., '*mis*conception', '*alternative* framework', or '*naïve* conception'). However, when we observe children in action – such as when Jane, Melissa, and Sylvia explore their mystery object and build models thereof – there is nothing that should lead us to the sense

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[36] Jean Piaget and Bärbel Inhelder, *The Child's Conception of Space* (London: Routledge & Kegan Paul, 1967), 22.

[37] Piaget and Inhelder, *Child's Conception*, 24.

[38] Maurice Merleau-Ponty, *Le sens et le non-sens* (Paris: Gallimard, 1996), 104.

of a lack. Learning and development is not the filling of a conceptual or cognitive void, but an engagement with the world as given to the children in their experience. The «cube» in Melissa's hand is no less real than the «rectangular prism» in Jane's. Moreover, Melissa's hand, too, comes to experience a «rectangular prism». But it does so not because of a lack. Rather, the empirical materials show that there is an excess of the new (plenitude) over the old plenitude, which, up to that point, was sufficient in and to itself. Learning and development then is the result of the realization of a possibility (perceiving the mystery object as a «cube» from not perceiving it at all; perceiving the mystery object as a «rectangular prism» when so far having perceived it as a «cube»).

    The possibility of this new is already contained in what was before and is born from it. The 'more' in the movement as Vygotsky thought of it allows the tactile perception to overcome itself. Sublating, in the dialectical tradition, means both doing away with and keeping something. In the constructivist literature, we can frequently read that misconceptions (here the «cube») have to be eradicated. In a phenomenological appreciation of the learning process, on the other hand, the sense-giving transition that corresponds to the phenomenalization of the new is sedimented in what is born; but as sedimentation, it also constitutes a forgetting. We know from a subsequent conversation with the teacher that she wanted Melissa to 'construct' 'on her own' a rectangular prism (ideally, as embodied mental structure and materially, in plasticine). Constructivist researchers, much in the way Piaget had done, focus on children's (learners') errors and faults, the *mis*conceptions that they have about some aspect of the world presented to them in lessons or clinical interviews. However, the «cube» did not arise as the result of a misinterpretation or a default. It was born in Melissa's originary perception, in and as sensual experience, intimately joined to her exploratory hand-arm movement. Those very capacities that led to the initial birth of a «cube», the realization of a cube from its possibility, are the same that also led to the emergence of a «rectangular prism», and, with it, to the destruction / disappearance of the «cube». That new experience is the possible but unseen – signified by the Vygotskian 'more' that is in excess of movement – and the encounter of the familiar with the alien. The possibility of the new is contained in the present even though it also means an overturning. The Spinozist-Marxian position presented in this chapter overcomes the problems of these other approaches 'because it immediately excludes any possibility of treating it in a vulgar materialist, mechanistic key, i.e. of identifying thought with the material processes that take place within the thinking body (head, brain tissue)'.[39]

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[39] Il'enkov, *Dialectical Logic*, 35.

3

The Mathematics of Mathematics

'In the ways that theorem provers prove theorems, they exhibit the adequacy of their ongoing work for proving theorems'.[1]

The going approach to the role of the social is to mark anything that happens when a group of students do something together (alone or with the teacher) as 'social construction'. Often, as in the widely written-about 'socio-mathematical norms', groups are said to 'negotiate' norms prior to acting according to them. In this approach, however, the social is taken in a trivial sense. Following the Vygotskian diction that any higher psychological function attributable to an individual *was* at one time a real societal relation with another person, I develop in this chapter a very different approach, where the genetic origin of mathematical reasoning exists in a transactional order rather than in people. Here *transaction* means that human activity cannot be reduced to individual actions; instead, individuals contribute *parts* to inherently social actions that can be understood only in their relation to the whole. Thus, whatever the provers of mathematical theorems do, whether individually or with someone else, they exhibit in and to the public, as the introductory quotation states, the adequacy of their ongoing work. It is because they exhibit this adequacy that others can see it as mathematical. Moreover, it is precisely because of the adequacy in each step of a proof that the proof can actually be achieved. That is, each practical step in the proof procedure maintains the conditions for acting mathematically so that were a proof to be done in the presence of others, they could see the current order and could take a turn in the prove procedure. The primacy of the social has to do with the fact that order production proceeds such that both order and the work producing it are visible. Precisely in this aspect, in this production of the visible order such that it can be perceived – whether or not someone else is actually present – precisely herein lies *the social* as *the irremediable mathematics of mathematics*. In this chapter, I focus on the social character of mathematical practice; in chapter 4, using materials from a second-grade mathe-

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[1] Eric Livingston, *Ethnographies of Reason* (Aldershot: Ashgate, 2008), 201.

matics classroom, I show how this orderly way of doing mathematics, in the life of an individual, first *was* a social relation with another person.

Late in his life, Vygotsky was turning toward understanding human activity through an analysis of the whole person. He no longer attempted to understand thinking on its own, as if who we are and what we do was the result of brain activity. Thus, he emphasized that one cannot make sense of the activity of *any* nervous apparatus without understanding the person as a whole. A brain always is the brain of a human being, not something that exists for itself. Vygotsky continues to reflect that the man, the person, is not the soul, is not mind. But what is a *human* being? He answers, '*the social personality of a person. A person as a member of a specific [societal] group.* As a definite *social unit.* A*s a being in itself – for others and – for itself*'.[2] It is precisely on this 'for others' that the present chapter focuses, because 'the social' is to be found here. The person, for Vygotsky, is a *social* unit, is a member of a group within society. He thereby follows Marx, as he states elsewhere in the fragmentary text entitled 'Concrete Human Psychology', in assuming that 'the essence of man' exists in the concretely lived, produced and reproduced societal relations that also come to exhibited in individual behavior. Whereas biology can explain the natural features of human behavior – e.g. the specifics of the vocal tract that afford speech, or the particular construction of the hand that affords specifically human movements – the necessity of labor, as Marx and Engels pointed out, is not included in the natural world and natural connections as B. Spinoza and L. Feuerbach had considered them. That sensuous, human labor children first experience as relation with another person; and as part of that sensuous labor, its very order is made to stand out with the help of sound-words that in this constitute the *accented* visible. This, then, is a core aspect of the present chapter.

Historically, geometry evolved from everyday non-geometrical practices of the early Greek. Rather than being opposite to formal mathematics, the intuitions of the pre-geometrical Greek constituted the *foundations* and possibilities for formal geometry to emerge. In this chapter, a distinction is made between (a) the subjectively experienced, lived work of mathematics and (b) the formal, objective written accounts of mathematics. The two are irreducibly tied together to produce geometry as objective science. Without the living, sensuous labor of *doing*, there is no geometry; without the written accounts, geometry could not be handed down as *objective* science: 'Documented in writing, the ideal object exists virtually "in the world", subject to actual creation at any instant'.[3] Together, the two aspects constitute a Lebenswelt pair.

Most research on mathematical experience does not tend to heed this distinction, focusing instead on its second part (b). In this chapter, I attend to '*rediscovering and exhibiting the naturally accountable mathematical proof, in its identifying detail for mathematicians, as a social achievement*'.[4] The text describes the struc-

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[2] Lev S. Vygotsky, 'Concrete Human Psychology', *Soviet Psychology* 27 no. 2 (1989), 64.

[3] Edmund Husserl, 'Die Frage nach dem Ursprung der Geometrie als intentional-historisches Problem', *Revue internationale de philosophie* 1 (1939), 212.

[4] Eric Livingston, *The Ethnomethodological Foundations of Mathematics* (London: Routledge and Kegan Paul, 1986), x–xi.
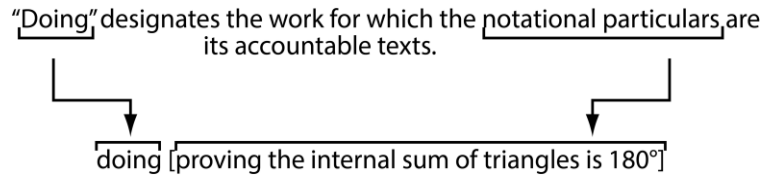
Fig. 3.1 Any social action involves sensuous work and that work's accountable text, a relation that also has been referred to as 'Lebenswelt pair'

tures of the lived work of proving and to exhibit the transcendental character of this work and of the objects that are the integral and constitutive parts of the resulting proofs. The purpose of the chapter is to articulate in an accessible way the primacy of the social in mathematics, which is in fact *the mathematics of mathematics*. I insist on the difference between (a) lived, sensuous mathematical experience, mathematical praxis and (b) *accounts* of mathematical experience. Almost all research, both quantitative and qualitative, is concerned with the latter, the accounts of mathematical experience, rather than with mathematics as actual, lived, sensuous praxis. I articulate the difference between the two and provide some guidance with respect to the ways of going about researching the lived work rather than accounts thereof. In this, I counter the false belief that our experiences are 'constructed', and insist that the real work that makes mathematics an objective science is actually lived and the result of our living/lived bodies rather than that of the constructivist mind. Whereas that which stands out in human experience is subject to intersubjective differences, that is, that which is 'constructed' and 'negotiated' in culture, the very nature of the sensuous human labor that produces the experience, the internal and invisible life that actually brings about experience, derives from the common structure of Being and the thinking body. In this manner, I state an approach that is an incommensurable, asymmetrical alternate form to *formal analyses* of living/lived mathematical experience specifically and of living/lived experience more generally.

Social Action: Work and Accounts

The specifically social nature of praxis derives from the way in which the sensuous labor of doing is linked to (verbal, written) accounts thereof (Fig. 3.1), which I articulate here for the mathematical praxis of proving exhibited in the subsequent section. Thus, the expression 'doing [proving the sum of the internal angles of a triangle is 180°]' consists of two parts. The text between brackets topicalizes a particular practice that social scientists and educational researchers might be interested in; the text may appear as a *gloss* of what a researcher or lay participant might say that is happening. For example, observing a student, a teacher might explain to the researcher visiting the classroom that the former is 'proving the sum of the internal angles of a triangle is 180°'. This text is the account for what is cur-

rently happening. Similarly, if asked by the researcher what she has been doing, the student might gloss, 'I was proving that the sum of the internal angles of a triangle is 180°'. Almost all research in the social sciences and education is of this kind; this kind of research sometimes is referred to as *formal analysis*. Research methods are provided in articles to articulate how the researchers arrived at identifying the structures that appear between the gloss marks (i.e., between '[' and ']'). But formal analysis does not capture the first part of the expression: it misses the sensuous labor of actually 'doing' what the text describes. The content of the brackets, precisely because it *describes* what someone does, exhibits the absence of the lived, sensuous work that actually produces the proof. This moment of the expression allows us to ask the research question, 'What is the work for which "proving the sum of the internal angles of a triangle is 180°" is that work's accountable text?' or 'What is the work for which "proving the sum of the internal angles of a triangle is 180°" is that work's proper gloss?'[5]

   We may actually understand the difference between the two orders, the one created by formal analysis and the one produced by people as part of their everyday life in terms of Paul Ricœur's notions of mimesis$_2$ and mimesis$_1$.[6] The concept of mimesis$_1$ pertains to the facts that human beings develop a fundamental sense of how the world works as part of their lives; and these order-producing aspects of life prefigure their observations and descriptions. This point is foundational to ethnomethodology but is not recognized at its full value in other forms of research that devise and rely on *special* ('scientific') methods that are marked as different from human pre-understandings and ways of making and seeing the orderliness of the world. A world characterized, for instance, by dialogue and transaction, makes the lived world such that any event transcends the individual intent and understanding. Configuration (mimesis$_2$) pertains to the ways in which actions appear in the world of text that no longer is characterized by the temporality of the lived-in world. Instead, textual descriptions are reigned by specific plots and actions that are reduced to individual agents. At this level, the narrative is not purely in terms of pre-understanding but also involves explanation. 'Meaningful action is an object for science only under the condition of a kind of objectification that is equivalent to the fixation of discourse by writing'; and any such 'objectification is made *possible* by some inner traits of the action that are similar to the structure of the speech act and that make doing a kind of utterance'.[7] But in this same process, the relational character of the world, its transactional nature, comes to be reduced to actions of agents and their *inter*-actions. In the emplotment of an event, the why, how, who, where, and when of action come to be represented together with narrative particulars that explain, depending on the context, past events in more or less

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[5] These are paraphrases with the particular application to the present topic. See Harold Garfinkel, and Harvey Sacks, 'On Formal Structures of Practical Action', in *Ethnomethodological Studies of Work* ed. Harold Garfinkel (London: Routledge & Kegan Paul, 1986), 173.

[6] Paul Ricœur, *Time and Narrative Vol. 1* (Chicago: University of Chicago Press, 1984), 54–70.

[7] Paul Ricœur, *From Text to Action: Essays in Hermeneutics, II* (Evanston: Northwestern University Press, 1991), 150 and 151.

theoretical terms. At this level, the narrative forms and the requirement for intelligibility impose constraints on how events can be told.

In contrast to constructive formal analysis, ethnomethodology is interested in specifying the work by means of which the order of the everyday world, its structure, is produced and is accounted for and glossed by the bracketed texts. In other words, the question ethnomethodology pursues is that in the sensuous living/lived work, for example, of proving that the internal sum of a triangle (on the Euclidean plane) is 180°. Once we know the organization of the living/lived work, we will be able to predict the kinds of results people produce. However, from knowing the accounts, we cannot infer the nature of the living/lived work. This is so because accounts leave out all the ways in which actions fail to produce what people intend to produce. For this reason, phenomenological and ethnomethodological accounts of mathematics are related to formal analyses – whether quantitative or qualitative (e.g., phenomenography) – in asymmetrically alternate ways. This is not to say that ethnomethodology disputes the accounts provided by formal analysis; those achievements can be demonstrated and are demonstrated in and as of the outcomes of the living/lived work of doing mathematics. This asymmetry is radical and incommensurable, but nevertheless obtains to related aspects of mathematics. Ethnomethodology (as phenomenology) is not in the business of interpreting signs that people produce. Rather, its 'fundamental phenomenon and its standing technical preoccupation in its studies is to find, collect, specify, and make instructably observable the endogenous production and natural accountability of immoral familiar society's most ordinary organizational things in the world, *and to provide for them both and simultaneously as objects and procedurally, as alternate methodologies*'.[8] The examples I use here constitute such materials that allow readers, in and through *doing* the work specified, to experience *for themselves* the living/lived, worksite-specific (inherent lived) praxis of *doing* mathematics.

The upshot of this approach is that no account can get us closer to the actual living/lived experience of doing mathematics, even when and precisely because persons retrospectively talk about their living/lived mathematical experiences. This is so because these accounts inherently involve *re*presentations of the experience, that is, means of making some past experience present again. We do not get the experiences themselves. In any instance imaginable, these representations – the means of making a past presence present again – are different from the living/lived labor in the living present. Only metaphysics will make a claim to the contrary, because it has not recognized that ever since the Greek antiquity, scholars have attempted to access once-occurrent, sensuous *Being* in and through externalities, that is, things (beings, representations). Being (capital B) and things (i.e. beings, plural and small b) are not the same thing, though in metaphysical accounts of knowing and learning (which includes all forms of constructivism from Kant to the present day), the latter are freely substituted for the former. This led Marx to his critique not only of idealism but also of all materialisms before him, which do not recognize 'real, sensuous activity' so that they consider 'only theoretical behavior

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[8] Harold Garfinkel, 'Ethnomethodology's Program', *Social Psychology Quarterly* 59 (1996), 6.

as the truly human, whereas praxis is grasped and fixed only in its dirty-Jewish manifestation'.[9] This led Feuerbach, the materialist philosopher, to fail grasping sensuousness as 'practical humanly sensuous activity'. He grasped what is human only as abstract 'species' somehow inherent in the individual when in fact, for Marx, '*human* essence' exists as 'the ensemble [sic] of societal relations'.[10] He concludes that if sensuousness is not grasped as praxis, then the most we can get to theoretically is the individual as distinct from society. Societal life fundamentally is sensual practical; and it is the practical, sensuous nature of work and human relations that make life – and therefor the activity of the thinking body – societal in its essence.

Proving Work is Social because Practical and Sensuous

In the following description of mathematical practice, using proving as an example, I follow the kind of studies produced in the field of ethnomethodology of mathematics.[11] Such studies are concerned with the irreducible relation of living and lived work and accounts of this work, which is social through and through. These descriptions are consistent with the phenomenological studies of the foundation of mathematics (geometry), which recognize the co-presence of lived (subjective) and formal (objective) dimensions of mathematics.[12] Accordingly, there are records and accounts of mathematical proofs, on the one hand, and the living/lived labor of doing a proof, on the other hand. I take this approach because it is consistent with a Vygotskian take on the primacy of the social.[13] It is here that we find *the mathematics of mathematics*. But we find it only in the actual doing of it – so I encourage readers to produce the proof for themselves.

The Proof Account

The proof that the internal angle sum of a triangle is 180° involves the following. In a first step, we note the relationships between angles that are produced when a line crosses two parallel lines (marked by '»', Fig. 3.2).

a.   The pairs (,), ( ), ( ), and ( ) are known as corresponding angles; corresponding angles are equal (i.e., = , etc.).

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[9] Karl Marx, 'Thesen über Feuerbach', in *Werke Band 3: Die deutsche Ideologie* (Berlin: Dietz, 1978), 5.

[10] Marx, 'Feuerbach', 6.

[11] See, for example, Livingston, *Foundations;* and Livingston, *Ethnographies*.

[12] Husserl, 'Ursprung der Geometrie'.

[13] See Wolff-Michael Roth, 'On the Social Nature of Mathematical Reasoning', *For the Learning of Mathematics* 36 no. 2 (2016): 34–39.

**Fig. 3.2** The angles produced when a line crosses two parallel (») lines
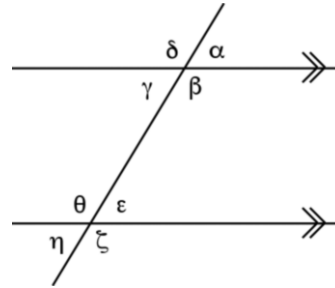
b. The pairs (   ), (   ), (   ), and (   ) are known as vertically opposite angles; vertically opposite angles are equal (i.e.,   = , etc.).

c. The pairs (   ) and (   ) are alternate angles. Alternate angles are equal (i.e.,   )
– because of (a) and (b).

   With these identities in place, we can prove that in the Euclidean plane, the angle sum in a triangle is 180° – if the total angle around a point is defined as 360°. This proof includes the following steps together with the diagrams in Fig. 3.3.

a. Any triangle can be drawn such that the base lies on one of two parallel lines and the opposing vertex on the other (Fig. 3.3a).

b. We know that alternate angles are equal (Fig. 3.3b).

c. Hence, because of configuration of lines at the upper parallel, that α,β, and γ add up to 180°, that is, α β + γ = 180° (Fig. 3.3c). Therefore three angles in a triangle add up to 180°.

   The preceding steps and figures do not constitute the entirety of the proof; rather, they constitute what we know to be the proof account. These are the parts that one might find in a textbook on geometry, on a website, or, in the case of new mathematical discoveries, in relevant journals. This is the part, therefore, that allows re-doing the proof over and over again. In that written part also exists the social of the proof: Anyone can re-do it recognizably, anywhere and at any time. This certainly has been done so since some time in antiquity, when the proof was accomplished for a first time. For example, the reviewers of an article take the submitted proof as instructions for doing the proof, checking whether there are 'no holes' in the proof procedure. In written form, this account suffices to be able to hand the proof procedure down – initially, to share it with others in the prover's community. But the proof, mathematics, does not exist in this procedure. The written procedure and diagram is not a cultural object. Instead, the mathematics of mathematics exists in the pairing of doing the proof and the account thereof. It is only in this way that mathematical proof is alive. Ordinarily, newcomers to a discipline learn the praxis of this prove in face-to-face work with others where monitoring and feedback correct actions; but the written accounts are such that they allow others to re-discover the proof in their own praxis. This possibility for the rediscovery of the proof in fact constitutes the objective and tradable nature of geometry as objective science. Thus, writing enables the *continual objectivity* of ideal enti-

Fig. 3.3   Steps in and part of the account for the proof that the interior angle sum of a triangle is 180°
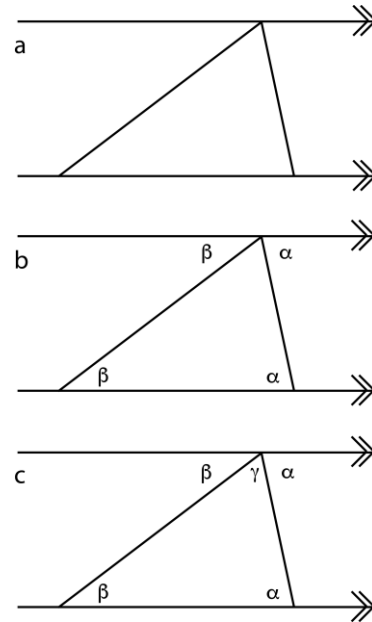
ties, as a form of shadow – the suprasensible that goes with the sensible object, as the Spinozist Marx would say. The ideal (subjective) objects exist virtually in the world in written form, and they therefore can be actually produced at any instant in time and wherever a person is located in this world. The lived labor (praxis) within which this written account *counts* as the proof, however, is not contained in the written account. *It is precisely this lived work that we are interested in here and in ways of capturing it*. To bring this proof to life we actually need to do it in and as of sensuous labor for which the written record has to provide sufficient resources.

The Sensuous Labor of Proving

The sensuousness of the labor of proving exists only in the doing. If it were represented in the mind, it would no longer be the sensuousness that comes with the actual doing in the same way that there is a difference between actually walking on a narrow beam and reflecting about walking on the beam. Moreover, the thinking in the course of the doing will differ from the thinking during reflection *on* the doing. I am interested here in the sensuous labor within which such accounts constitute the resources that allows us to count what is happening as a proof. In the present instance, for example, this living/lived labor includes the re/cognition that pairs of corresponding, opposite, or alternate angles are equal. That these pairs of angles are equal presupposes the seeing of each angle (see chapter 1 on seeing lines and figures). Such *seeing* is related to the sensuous work of drawing multiple

Fig. 3.4 In the dynamic of drawing a line, the plane becomes bisected, here denoted by a hatched and an unhatched part

a

b
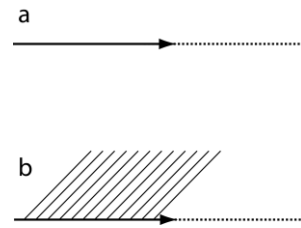
Fig. 3.5 Two intersecting lines produce four sectors

a

b
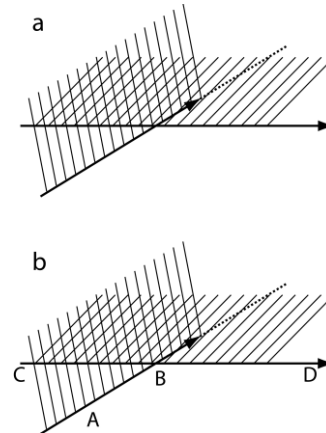
lines, each of which bisects the plane (Fig. 3.4). This work involves particular movements, 'integral kinaesthetic structures or kinetic melodies'[14], which are inscribed in the living/lived body (the flesh). From the perspective of the living/lived work, the writing gesture produces the divisions of our pre-geometrical experience of left/right, up/down, and so on. When, after the completion of the first line (involving a complete bisection of the plane), a second line is added, it, too, bisects the plane. Four sectors are thereby produced, which appear in three different hatchings: not hatched, hatched once, and hatched twice (Fig. 3.5a).

We could have also drawn the second line in the reverse and produced the same account. For this reason, the angles enclosing the single-hatched areas are the same. What is in the first drawing the angle forming first to the left and then to the right will be, upon beginning the diagram from the other side, again first to the left and then to the right. In this very act of drawing, we also produce an order that goes with the naming of locations (Fig. 3.5b). In this way, the unfolding from the drawing the AB line with respect to CD forms angles ABC and ABD, which we may also name, following the tradition, by the Greek letters α and β (as well as the equivalent angles γ and δ) (Fig. 3.6). Here, the order in the actual making constitutes a conceptual order: 'The temporally placed label of an angle or its apparently disengaged placement in a finished figure exhibits this seen relationship as a proof-

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[14] Aleksandr R. Luria, *The Working Brain* (Harmondsworth: Penguin Books, 1973), 176.

Fig. 3.6 The placement of the labels a and b is apparently disengaged from the temporal practice of drawing the figure
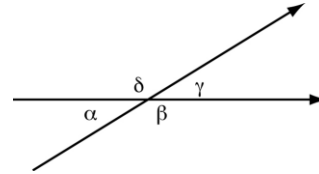
specific relevance'.[15] The conceptual order is *in* and *arises from* the sensuous movement – i.e. 'sensuousness as the *practical*, human-sensuous activity' of the Fifth Thesis on Feuerbach[16] – rather than from the constructive mind, if something of that kind indeed exists.

The relationships between the lines, angles, bisectors, and sectors have to be seen; this seeing, as shown in chapter 1, is based on the movements of the eyes, movements that we are not in conscious control of. Not surprisingly, phenomenological philosophers have recognized the fundamental passivity that is associated with a first cognition that such seeing involves – the same kind of first cognition that Spinoza highlighted when stating that we follow the contours of Nature with our body movements. Any first formation of sense has two passive moments: the first existing in the first cognition, the second in the fact of the retention of this first cognition. Thus, 'the passivity of the initially darkly awakened (insight) and the eventually increasing clarity of that which appears is accompanied by the possibility of a change in the activity of a *remembrance*, in which the past experience is lived again actively and quasi anew'.[17] The memory is awakened passively but can be transformed back into the corresponding sensuous labor. The *re*cognized relationship may therefore be maintained throughout the proof procedure, which leaves as its end result the sequence of diagrams (Figs. 3.2, 3.3). In the drawing, we do not specify a particular angle to be produced. Any labor that produces two non-parallel lines suffices to get us to this point. This fact produces the generality of the proof procedure.

This memory is important in the constitution of geometry as an *objective* science in and through the subjective, living/lived labor of the geometer. A sense-forming act that came about spontaneously can be actively/passively remembered, and therefore reproduced not only by the original individual but by any other individual as well. It is in the reproduction of the living/lived work that the evidence of the identity between original and subsequent act arises: That which the subject now reconstitutes, but does so in an original act, is the same what had been evident before to someone else – geometry is reproduced, over and over again, originally, in the act of everyone doing geometry. That is, together with the original sense formation comes the possibility of an arbitrary number of repetitions that are identical in, and as of, the chain of repetitions. That is, the very living/lived labor that allows me to recognize relationships again make for the social nature of geometry and its historicity as objective science.

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[15] Eric Livingston, *Making Sense of Ethnomethodology* (London: Routledge & Kegan Paul, 1987), 96.

[16] Karl Marx, and Friedrich Engels, *Werke Band 3* (Berlin: Dietz, 1978), 6.

[17] Husserl, 'Ursprung der Geometrie', 211.

Interestingly, the very generality of the proof – the mathematics of mathematics – derives from the way in which the sensuous labor unfolds. This work is not mine, not something individually constructed. It is recognizable by others, who, in this, recognize a move that they might have conducted themselves and that therefore – in this recognition – is social through and through. For example, in the drawing of a line that crosses two parallel lines and labeling alternate angles using the same letter, the proof makes available that any such line could have been drawn, which in fact occurs when the second line between the two parallels is drawn such as to form a triangle. The very possibility to have one line between parallel lines with alternate angles enables all other lines. The relations between the angles in configurations of parallel lines crossed by a third thereby imply the specific value of the angle sum of the triangle (i.e. 180°). That is, from the way in which living/lived labor produces parallel lines and the equivalent angles that follow from (the idea of) parallelism simultaneously constitute the angle sum to be 180°. That this is so can be discovered over and over again because (necessarily written) proof-accounts describe their own work. That is, it is precisely 'in this particularistic way, [that] the generality of our proof-account's description was evinced in and as the lived, seen, material details of the proof'.[18] The very nature of geometry as objective science arises from the demonstrability and visibility of its procedures in the living/lived (subjective) work of proving. Anyone may reproduce the living/lived labor anywhere. In sum, therefore, we realize that the generality of the proof exists in and simultaneously does not exist in the proof-account. The proof lives in the pairing of an account with the lived work that produces the account. That is, it lives in the continued transformation of the account into labor, and of labor into accounts. With this, we are at precisely the conclusion that the most outstanding of Spinozist-Marxian thinkers has arrived when he states that

> 'The ideal form is a form of a thing, but outside this thing, namely in man, as a form of his dynamic life-activity, as goals and needs. Or conversely, it is a form of man's dynamic life-activity, but outside man, namely in the form of the thing he creates, which represents, reflects another thing, including that which exists independently of man and humanity. "Ideality" as such exists only in the constant transformation of these two forms of its "external incarnation" and does not coincide with either of them taken separately'.[19]

Il'enkov here writes about the ideal in general, though he discusses in this article mathematics as one of the specific domains. The 'labor' in my account is Il'enkov's life-activity, and his 'thing' corresponds to any of the proof accounts provided above. In the Spinozist-Marxian take, therefore, does not exist in 'mathematical representation' nor in 'mathematical practices', but in the constant transformation of labor into material form and of material form into labor.

In the preceding analysis, we see how the living/lived work of producing, seeing, and labeling the angles is actually accomplished. This drawing, seeing, and

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[18] Livingston, *Foundations,* 108.
[19] Evald V. Il'enkov, 'The Dialectics of the Ideal', *Historical Materialism* 20 no. 2 (2012), 192.

labeling is available to those present; this drawing, seeing, and labeling makes the work objectively available to those present – for others and, in this, for myself, just as I quote Vygotsky above. But this living/lived sensuous labor does not (and cannot) appear in the proof account proper, where the lines and labels appear disengaged from the actual drawing, seeing, and labeling. All of these involve our living, that is, thinking body. Seeing an angle involves fewer lines, but nevertheless requires the movement of the eye that puts into relation the two unfolding lines, the half planes, and the seeing of the intersecting planes against the background (generally white). That is, the materials in the account guide the eyes to see something specific, and it is as such guidance that we can speak of the transformation of Il'enkov's 'material thing' into labor. The account that we might find in textbooks is disengaged from this *sensuous* labor, but it may serve as a resource on the part of the learner, to relive the sensuous labor of proving in and through his/her own movements of drawing, seeing, and labeling. There is a relation between accounts and the sensuous work: In textbooks the actual production of the primal geometrical idealities is surreptitiously substituted by means of drawn figures that make it possible for concepts to be sensibly intuited. It is up to the students to find in their own sensuous labor the practical relevance of what the account describes, which, in the present example, would be the proof-specific relevance of the lines, markings, naming, and so forth.

We can see that in this pairing of proof account and sensuous labor of proving there is the possibility of pedagogy. In fact it has been said that the proof account also is a pedagogical object in that it can function to teach the lived-work that it describes – recognizing that the relationship between account of lived work is like the relation between a cooking recipe and the work of cooking that produces the edible dish. This is so because we can see in it a *formulation* of the labor that is described, much like an instruction that presents both what is to be done and an image of what will be found as an outcome of the actions. However, this condition still does not solve the ultimate problem of the difference between the account and the sensuous labor that we actually feel: the students have to find in their own living actions the relevance of this or that definition, this or that instruction, this or that description of an outcome. There is a surplus in the transitivity of the sensuous labor over its ideation that constitutes the difference between sensuous work and any account thereof. In a personal note from the last year of his life, Vygotsky focuses on the importance of that surplus, which he refers to as the 'invisible sense of movement'; this invisible sense arises from 'movement [that] always has a latent, inner sense of movement, which always expresses the person's attitude to the goal, the internal obstacle, struggle, hesitation, additional goal, latent tendency or motivation, hot temper, weakness, exaggeration of the goal, attainment of the goal for show, etc.'.[20] The visible sense, its orientation toward the goal, makes movement comprehensible and, therefore, social through and through. But the invisible sense makes it personal, sensuous. The order produced in and by the thinking body therefore is neither just individual nor just social: it is mine and not mine simultaneous-

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[20] Lev S. Vygotsky as quoted in Zavershneva, 'Notebooks', 53.

ly, individual and social. It is me doing the labor but in a way entirely comprehensible on the part of the other because *my* labor is *for the other*. In this 'for the other' is buried an irreducible connection between the two; and another name for 'irreducible connection' is *transaction*.

In this section, I articulate but the beginning of an analysis that indicates the nature of the sensuous labor as distinct from the objective accounts produced and handed down for millennia from the ancient Greek to the present day. The accounts are objectively available to all the generations; the subjectively lived, sensuous labor has to occur each and every time someone is actually doing or following (observing) the proof. In this way, the subjective labor of geometry and the objectively available account are intertwined to make geometry the objective historical science that it is. The sensuousness of labor has nothing to do with a mental construction, as the movements underlying the (intentional) drawing of a line emerge from experiences that have nothing at all to do with intentions. These are originary movements that have nothing do with the '(embodied) image schemas' of cognitive science and embodiment/enactivist accounts. But they may indeed be thought of as *archetypal corporeal-kinetic* forms or as kinetic melodies that would enable any such schemata, if they existed at all.

Making Sense of Graphs: The Primacy of the Social

'For a sociology of the witnessable order to make any sense, it must begin and end with the observable work of the ordinary society'.[21]

The previous section shows how *proving* exists in and as sensuous labor, which is produced by, and thereby affects, the thinking body and is felt affectively. Precisely in being sensuous, it is also social, marked as such by the proof account, which also may serve as instruction for others to bring the proof to life through their own sensuous work. The description focuses on the proof and how it may be lived and relived again and again. In this section, an example from another area of interest to mathematics is provided: reading and interpreting graphs. Rather than drawing on the many think-aloud protocols of scientists reading and interpreting graphs that feature in my database, a pair of scientists is featured here in a fragment from one part of a task. In showing how participants attend to the irreducibly *joint* labor of graph reading, in the details of their joint rather than individual labor, the analyses exhibit the intrinsically social rather than accidental (because there are two) nature of reading graphs. In this situation, the conditions of the joint work are produced, monitored, and maintained.

In constructivist (constructionist) research, arguments are made and data used to exhibit the apparent contingency of knowledge claims and practices. This contingency is said to derive from the effects that culture, race, gender, relations, politics,

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[21] Livingston, *Ethnographies*, 207.

finances and the likes have on whatever it is that counts as knowledge and fact. In this situation, however, 'the problem remains that the social character of domain-specific skills and reasoning may only be an incidental feature'.[22] In contrast, the claim of a primacy of the social is that something like a scientific reading of a graph is in itself a social thing so that its social character is neither accidental to doing of a graph reading but is present to practitioners as the produced and witnessed reading of a graph. This is the sense Vygotsky articulates when he writes that 'real relations between people underlie all higher functions and their relationships' so that – given certain conditions – we may observe a '*renewed division into two of what had been fused into one*'.[23] Such conditions include situations where (a) children participate with others who already embody these higher functions and their relations and (b) some form of trouble arises in the joint labor of two or more individuals already embodying those same higher functions and their relations. In cases where scientists do not have an immediate pat answer for a given task, they often have to articulate for one another what are otherwise invisible aspects of their labor.

Before getting to the actual case, a description of the relationship between text and reading is required, where 'text' is a general term that subsumes not only figural materials – often referred to as 'external representations' or 'inscriptions', depending on theoretical approach – but also architectural features or regularities in nature, such as foot prints that are read by (aboriginal) hunters. Even actions can be subsumed to the paradigm of the text. Text and (competent) reading come as a pair, as an account and the associated work. Thus, 'the work of reading is hopelessly intertwined with its text, and reading's work is the work of finding how a text describes its reading'.[24] When reading a text – such as the graphs in the present situation – is conducted with multiple participants, then their talk, which is a protocol of the joint work to arrive at a common statement about what the text says, tends to articulate both the properties of the text (here graph) and aspects of the work ('finding how the text describes the reading'). These relations are captured in the reflexive proposition that 'the work of reading is the work of finding the organization of that work hat a text describes'.[25]

Fragment from a Graph Reading Session

The two participants were PhD students (Lori, Jon) in particle physics, both within a few months of their graduation and both now working in a university setting. They had been invited to participate in a study of expertise in graphing within and out of field. As all other participants, they were presented with three sets of line

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[22] Livingston, *Ethnographies*, 212.

[23] Vygotsky, 'Concrete Human Psychology', 58.

[24] Eric Livingston, *An Anthropology of Reading* (Bloomington: University of Indiana Press, 1995), 14–15.

[25] Livingston, *Reading*, 14.

Fig. 3.7 Jon and Lori work on a graph interpretation task that includes this text: 'Many systems, one variable such as temperature) depends on a number of factors, for instance, the absolute change in the thermodynamic variables V_1 and V_2. Any single variable can limit temperature increase. Sometimes scientists study the effect of pairs of variables. The graphs depict three different physically realistic scenarios of how two variables (V_1 and V_2) might combine to affect temperature. Discuss the effects of different levels of the two variables on each amount of the third (20, 50, 80) in each scenario (a, b, c) (Draw as many inferences as possible)'
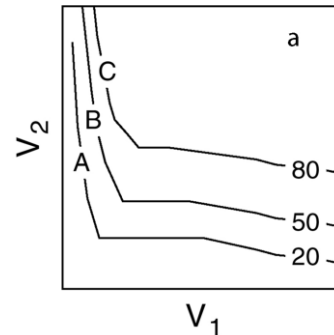
graphs from physics and three sets of graphs from biology that exhibited the same fundamental relationships. One of the three physics-related graphs consisted of a panel of three graphs, each of which shows the effect of two independent variables (v_1, v_2) on temperature, the third variable (Fig. 3.7). On this particular graph, the success rate of physicists doing the task individually was 24% when compared to the standard interpretation.[26] Lori and Jon sat next to each other, focusing on the same graph on their respective worksheets, following or marking with their pencils particular features (Fig. 3.8). For example, while saying 'for a given vee-two', Lori moves the pencil tip, which she has previously brought to a particular value on the v_2 axis, parallel to the horizontal v_1 axis.

Fragment 3.1[27]

```
01  J:   for a givn vee-two ((pencil on one value of v₂))
02       (0.5)
03  L:   right yea yes you're=right >for a given< vee-two: (0.4) above this little
         [            (0.2)          ]
         [[((moves horizontally to v₁))]
         [this little (0.4) n'diagonal >n set of juts<    ]
         [[((moves repeatedly over the three elbows))]
04       (0.7)
05       yea (0.1) [             °thats right°                  ]
                   [((moves over three 'elbows' in Fig. 3.7))]
         (0.2) temperature is virtually independent of vee-one
06       (0.2)
07  J:   right (0.3) °°hm°° (0.4) >so there is [some sort of phase transition] going
                                              [  ((moves rH as in Fig. 3.9)) ]
         on here. we have (0.1) >below< vee >below vee-one, we have some re-
         gime .hhhh >°so°< >below< some critical value of vee-one vee-one crit
         (.) .HHhhh (.) uhh (0.6) there is (0.3)
08  L:   >yea [for a gi]vn vee-two like< (.) [that]
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[26] Wolff-Michael Roth, 'Limits to General Expertise: A Study of In- and Out-of-field Graph Interpreta-
tion. In *Encyclopedia of Cognitive Psychology* (Hauppauge, NY: Nova Science, 2012), 311–348.
[27] The transcription conventions are available in the appendix.

Fig. 3.8 Jon and Lori working on a graph interpretation task as part of the graphing practices of scientists. The pencils *follow* and, thereby, also symbolize properties of the text

```
09   J:          [°°uh°° ]                          [yea ] °exactly°
 10       (1.1)
11        we have regime whereby
12        (0.6)
13   L:   >small changes[   in  ] [  tem ]perature make BI:G< °ch[ang]es in [crit°
14   J:                    [°tèm°] [°tèm°]                          [wa-]     [>lets-
          say< small changes [in vee-[one]             ]
                             [((hands, Fig. 3.11a))   ]
15   L:                               [vee]-one        ]
                             [((hands, Fig. 3.11a))   ]
16   J:   big [changes      ] in temper[ture  ]
             [((Fig. 3.11b))]
17   L:                                  [right]
          [((Fig. 3.11b))]
18        (0.8)
17        °thats what I meant°
```

In the course of this fragment, the two make stand out the elbows ('juts') of the three curves, which have critical values of v_1 axis identified as $v_{1,crit}$. The statements suggest that to the above (i.e. right) of these values, the temperature functions are 'virtually independent of vee-one' whereas below (i.e. to the left), 'we have a regime' where 'small changes in v_1 bring about 'big changes in temperature'. There are also a lot of hand movements, with or without the pencils, which *follow* – not 'construct'! – particular lines, both the ones actually part of the text and others projected onto it (e.g. the one marking a 'given vee-two'); and this following essentially also directs or aids the movement of the eyes and the making salient of these textual properties. There is an apparent correction ('let's say') to an initial statement about the relationship below $v_{1,crit}$ (turn 14 & turn 16), which apparently states 'what was meant' to be said (turn 17). All of these features are properties of the text that this reading takes into account, or discards, to arrive at a

statement of how the graph is to be properly read; and this reading requires a particular organization of the different properties identified. Thus, the 'jut' at the 'vee-one-crit' separates two 'regimes', one in which the 'temperature is virtually independent of vee-one' and one in which small changes in 'vee-one [are associated with] big changes in temperature'. Although all these words, 'critical', 'regime', 'temperature', 'diagonal', or 'change' are common in the English language, the ways in which they appear in the statements may actually not be so apparent so that non-mathematically savvy individuals may not follow the two physicists in what they are doing and grasp what they are talking about.

At the same time, all the words and hand movements that point or follow graph features have an apparent function: they accent the visible. It is in the use of words and hand movements that corresponding features in the graph are marked as visible. In this way, communication lets that be seen that communication is *about*. Referring to verbal communication, Heidegger calls this the *apophantic* function of logos as speech.[28] Communication has this function both for the speaker and the recipients, which orients us to the inherently social nature of communication. Thus, even in the case of the scientists doing the task alone, and even if the research assistant had left the room, pointing, following lines, and using words are social because they are directed towards the Self as an Other.

The Social in/of (Graph) Reading

The publication of books entitled *An* Anthropology *of Reading* or *Toward an* Anthropology *of Graphing* suggests that there are ways of approaching reading texts and graphs other than by attributing this process to events in the individual mind (brain).[29] The term 'anthropology' and the associated ethnographic process of collecting data suggest that the work of reading is observable. In fact, making this assumption immediately leads to the conclusion that children learn to read because of this very observability and the possibilities for participating in the work of doing reading. When people read on their own – an observation readily made in parks, on public transport vehicles, or in other public places where individuals are gazing at newspapers, tablets, or smartphones – this social character of reading is not easily apparent, though by means of a little work it can be shown that reading is in fact social rather than individual. As Vygotsky suggests, even 'to read one's own jottings' – e.g. in one's personal diary – 'means to relate to oneself as to another', that is, is a form of social labor through and through.[30]

We observe evidence in the preceding, brief fragment that speaking not only is for the other, but for the speaker as well. For example, following an acknowledg-

[28] Martin Heidegger, *Sein und Zeit* (Tübingen: Max Niemeyer, 1977), 28.

[29] Livingston, *Anthropology of Reading*; Wolff-Michael Roth, *Toward an Anthropology of Graphing: Semiotic and Activity-theoretic Perspectives* (Dordrecht: Kluwer Academic Publishers, 2003).

[30] Vygotsky, 'Concrete Human Psychology', 58.

Fig. 3.9 Jon's right hand moves twice in a diagonal fashion, here shown between the two extreme positions of the right hand, the first time less extended, the second time as shown

ment of the focus on a given (value of) v_2, articulated verbally ('given vee-two') and gesturally (movement of pencil parallel to v_1 axis), the three elbows in the graphs are made to stand out, again both verbally ('set of juts') and gesturally (repeated pencil movement across the feature, with the right hand following a 'diagonal' trajectory, as in Fig. 3.9). A long pause follows. It provides an opportunity for Jon to take the speaking floor; but he does not take it up. The pause also provides an opportunity for Lori to speak again, which she does by providing a commentary on what she has said before: 'yea, that's right'.

Turn 14 exhibits the monitoring, as do the repeated articulations of '(that's) right', 'yea', or 'yes'. In this, the monitoring itself is made available to the other – and it is well known that in cases where recipients do not articulate attending to what is being communicated, this absence itself may be made observable, for example, by saying 'are you listening?' or 'did you hear what I was saying?'

To understand the function of language, we must not look at it abstractly, asking about 'the meaning' of words and phrases. If we were to do so, then we would not go beyond inferences concerning something behind the words actually used and phrases actually said. The 'meanings' therefore are treated as invisible, existing somewhere in another, metaphysical world; and special research methods are therefore required on the part of the investigator to construct those invisible orders of the event. The constructivist approach begins analyses with the 'meaning' of an individual speaker, which may or may not be the same as the 'meaning' of each recipient. Vygotsky however points out – citing the philosopher of materialism Ludwig Feuerbach (see chapter 1) – that the word is impossible for one person and always is a reality for two persons. Thus, speech is central because it is both a social relation and a psychological means. This statement, which may appear cryptic to some readers, immediately makes sense when we considering consecutive turns such as turns 05 and 07 (turn 06 is a brief pause). The statement that concludes the first of the two turns, 'temperature is virtually independent of vee-one' does not just belong to Lori (Fig. 3.10). Instead, while Lori's vocal cords, jaw, and lips are moving, the parts of Jon's inner ear responsible for hearing also are moving. That sound, those same frequencies are common to both. The two are literally connected. From a physical perspective, there are two connected cavities in resonance.

The revised transcription exhibits precisely this co-presence of the sound in the mouth of one and in the ear of the other. Their relation is, in part, this resonance, a phenomenon understood only if we take into account the resonance cavities and the moving air within and between them. All the phenomena of interest to mathe-
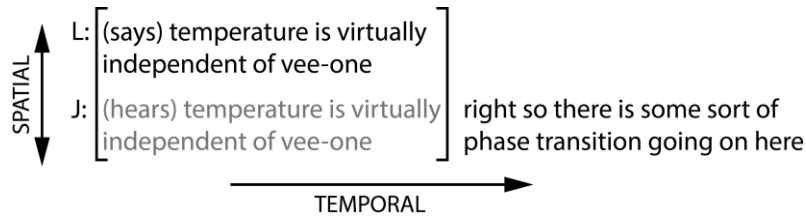
Fig. 3.10 In conversation, the sound-words are common (simultaneous) to speaker and recipient. The said, which affects the recipient, is the (external) beginning of his response that ends with the reply

matics educators, such as mind, consciousness, and (inter-, intra) subjectivity are grounded in the materiality of speech and other perceptual modalities. Thus, consciousness, mind, Marx and Engels say, is cursed by its material nature, which 'here makes the appearance in the form of agitated layers of air, sounds, in short, of language'; and 'language *is* the practical, also for other people existing, so therefore also for myself existing real consciousness'.[31] In and through the materiality of the *sound*-words, consciousness exists for the recipient, and, in this way, also for the speaker. Lori finds out what she said and marks something Jon articulated as what she has meant to say even though she has not said so.

Mutual attunement is made available not only by means of language but also in the coordination of body movements. Coordination means that there is a whole that exceeds the sum of the part – like the two clocks on the same wall that entrain each other so that they begin swinging with the same period, a phenomenon that would not be understood if the clocks were taken to function independently. The parts are parts of a system sui generis, and, therefore, need to be understood as parts thereof rather than as independent parts coming together so that their effects could be added up. The frequencies of the two cavities become synchronized with each other at a frequency different from their individual frequencies.

Even though the sounds between them are the same, Lori and Jon may hear the sound-words differently in the same way that Melissa and her two peers feel the mystery object differently in chapter 2. Indeed, Lori knows the effect of the words she has said only after perceiving the reply. This is apparent when the next turn to 'small changes in temperature make big changes in crit-' comes to articulate a different fact, 'small changes in vee-one, big changes in temperature'. The description 'that's what I meant [to say]' marks that what was said differed from what was meant to be said, and the latter indeed matches with the immediately preceding statement.

The retranscription of turns 05 and 07 (Fig. 3.10) thus brings to the fore that sound-words constitute a spatial dimension of communication, which connects rather than disconnects the participants in talk. Again, and to be emphasized here, language, words, and phrases are not considered in the abstract but rather always in their actual use by real people doing something real to them in the here and now of

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[31] Marx and Engels, *Werke Band 3*, 30.

Fig. 3.11   Coordination of hand movements with 'small changes in vee-one' (above) and 'big changes in temperature (below)

their situation. The question is, 'What are the participants doing with language?' The resonance phenomenon of which the two are part can also be seen directly in the corresponding body movements, which articulate perceptual aspects in an iconic (rather than verbal-abstract) manner. Thus, the articulation of 'small changes in temperature' goes together with closely positioned hands of both participants (Fig. 3.11, upper), much as larger hand separations on the part of both participants fall within the articulation of 'big changes in temperature' (Fig. 3.11, lower). Not only semantic aspects of the featured reading are in common – as per the formulation 'that's what I meant' (turn 17) – but also body movements are aligned; and so are the semantic aspects of the sound-words and those of the hand movements and relative positions.

The retranscription (Fig. 3.10) also makes apparent a temporal dimension. More specifically, when the question is, 'What did the statement Lori articulated in turn 05 do in this situation?', then the answer can be provided only when the effect of this action is known. This only happens when the reply is completed. That is, if we ask for an explanation of the unfolding of the event between just prior to the transcribed words in turn 05 to the end of the reply, then the cause is known only after the effect. This turns around the common understanding of cause and effect, where

the former brings about the latter, whereas here, the cause is known only after the effect. Already in the nineteenth century, it was noted that 'cause and effect [is] a dangerous concept when we think of some *thing* that *causes* and some thing *affected* by it', for 'necessity is not a fact but an interpretation'.[32] In the transcription, the response, which goes from active reception to the completion of the reply, the latter is given shape during the former part. But – as thought is completed in speaking and therefore thinking only knows itself through its own productions, that is, when speaking has come to a close – the affected comes to know its affection in that process, in the irreducible response, of which the reply constitutes only the second part. Thus, cause is revealed only over time and with the effect; and neither cause nor effect can be understood independently of all participants. Truly, then, the social has precedence over the individual – unless there were no relation at all, for example, if Jon had attended to something else or if the two were talking past each other. Thus, *if* there is a *con*versation, then the social (relational) has primacy over the individual.

At the outset of the analysis, I suggest that the work of reading not only includes the articulation of the properties of the text and their relations, which describe the reading, but also aspects of that work itself. The articulation of properties of the text, such as the 'juts' or 'vee-one-crit, can be understood as formulating the reading, that is, as stating what reading does (reveals). Other statements, such as the frequent uses of 'that's right', 'right', or 'yea exactly', make available to participants part of the joint work that produces the relation of Lori and Jon, in which agreement, a form of social order, is made manifest for one another. These are markers that the task-related labor can go on. So does the statement 'that's what I meant', which *formulates* alignment, that is, tells the other person that alignment exists. This formulating is part of the labor of articulating the consciousness (for other and self) that the activity is on route toward the common reading that was required as the task's goal. We observe Jon produce word fragments, as if wanting to speak until he eventually gets to articulate (turn 12) what can be heard as a correction of something said earlier. When Lori later says that she 'meant [to say]' was different (from what she had said), we find out not only that phrase was heard as a correction but also that Lori is in agreement with it, because it expresses what she meant to say.

Body–Mind Monism and the Primacy of the Social

In the course of chapters 1 and 2, I articulate a monist approach to the questions of the relation between body and mind or individual and society. One might therefore ask, 'How is it possible to speak of the primacy of the social all the while claiming to espouse a monist take?' The answer lies in the fact that the social is grounded in

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[32] Friedrich Nietzsche, 'Aus dem Nachlass der Achtzigerjahre', in *Werke in drei Bänden: Band 3* (Munich: Hanser), 540.

the body, as much as bodily phenomena are grounded in the social. The individual body and the collective social are but two manifestations of the same thing: living, *natural,* immortal society. Moving itself, however, is not particularly human; but the customary movements, because they have been shaped to a large extent by created ('cultural') things, are specific to humans – such as the movement trajectories of my hands that currently unfold over the keyboard to produce the words on this page. Unlike in other living beings, there is also a particular form of (verbal) consciousness that accompanies bodily movements; and this consciousness, though not necessarily making the movements thematic, reflects the same activity for the purpose of which the movements also are employed. Marx notes that 'man does not lose himself in his object only when the object becomes for him a *human* object or objective man. This is possible only when the object becomes for him a *societal* object, he himself for himself a societal being, just as society becomes a being for him in this object'.[33]

We observe how the traces on the paper (i.e. the textual properties) are, and are reconstituted as, societal and human objects especially in the fragment from the graph reading session. Through the labor consisting of moving and making sound-words, features of the material world come to stand out and are made present again. This doubled presence, a form of doubled experience, is specific to human beings and it allows humans to evolve forms of adaptation that are unavailable to animal species – though in its rudiments certainly existing already in primates.

It also appears evident that in speaking about the graph reading session it is inappropriate to speak of *inter*action, for there are not two independent actors, one acting after the other. As can be seen in Fig. 3.10, there are not two independent actions put together to create an interaction. Instead, the action of one requires knowing the action of the other: to know what the first phrase does *in this situation*, we do not need to go into Lori's head to find out 'her meaning' but we need to investigate what it has done. This we come to know from the subsequent phrase. Not only do we not need to get into Lori's head, but also Jon does not have to do so. This is so because everything required for moving this task ahead is made available right out in the public arena of their joint labor. If something required for getting the job done were missing, then someone would make it the topic of the ongoing talk. For example, Jon did not have to wonder what was in Lori's head when she began to say, 'below some critical value of vee-one'. Instead, when he was done talking, he had produced a statement with which she not only agreed but also indicated that she meant making herself. And to know the *why* of the subsequent talk, we do have to investigate the preceding phrase, the origin of which was outside of Jon's head.

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[33] Karl Marx, 'Ökonomisch-philosophische Manuskripte aus dem Jahre 1844', in *Werke 20* by Karl Marx and Friedrich Engels (Berlin: Dietz, 1975), 541.

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Sociogenesis

In chapter 3, it can be seen how mathematical reason is *inherently* rather than incidentally social. In this situation, Vygotsky suggests investigating those situations where something social – e.g. mathematical reasoning or mathematical classifying – first appears in the life of a person in her relation with another person. Vygotsky calls this method of analysis *sociogenesis*. We could also ask the question differently: 'How does mathematics reproduce itself?' If the mathematics of mathematics exists as the ensemble of societal relations, as shown in chapter 3, then the answer is much easier for readers less familiar with the Spinozist-Marxian approach inherent in this book. Thus, it is in and through societal relations that the mathematics of mathematics 'captures' another staff member who keeps mathematics alive. In other words, the social reproduces the social by enlisting new staff through sociogenesis. Psychologically speaking, I exemplify the notion of sociogenesis in this chapter 4 by showing how at some time in the lives of children, such forms of reasoning do not yet exist, when they first appear *as* relation with another, and finally when they show up as individual behavior. Sociologically speaking, I exemplify the notion by showing how a form of societal relation 'trains' a new staff, who then contributes to keeping this form of relation alive. Because mathematical reasoning first *was* a social relation, it can be unpacked into a social relation whenever required even after it has become an aspect of individual behavior. Examples when this unpacking occurs (a) when there is trouble so that participants request unpacking or (b) when teachers engage with students such that the complex behavior may show up as social relation again. Vygotsky realized that such unpacking occurs but never got to the point of actually describing and theorizing the phenomenon. The sociological subfield of ethnomethodology may serve as an example where and how to look: it seeks troublesome situations as part of the investigation because the generally invisible work of order production – e.g. of mathematical reasoning – here is made visible again.

Learning – From Perception and Construction to *Sociogenesis*

Existing accounts of learning and development, including its most advanced form in the *theory of knowledge objectification*, fall short of providing the kind of account that Vygotsky names *sociogenesis*. Research suggests, for example, that 'students' construction of a cognitive configurations network … [is] in line with the institutionally intended configurations'.[1] This suggestion is said to be equivalent to that obtained in the theory of knowledge objectification, where learning is said to consist in the form of an increasing subjective awareness of the 'cultural object', which, in the process, is endowed with 'meaning'. Individualization goes together with the absorption of the theory of knowledge objectification into social constructivism. This form of constructivism cannot capture the social because it is nothing but an extension of the self-actional model to an interactional model of human behavior. This model misses the primacy of the social at heart of the cultural-historical approaches generally and the theory of knowledge objectification specifically because here the social is taken to be transactional (Fig. 1.2).

In the Spinozist-Marxian approach, '*societal* consciousness', the (mathematical) ideal, 'a historically formed and historically developing system of "objective representations"', 'is not simply the individual consciousness repeated many times, just as the societal organism in general is not the individual human organism repeated many times'.[2] One affordance of Marx's presentation is that it allows apprehension of the distinction that he, as well as the cultural-historical psychologists that employ his method – e.g., Vygotsky, Leont'ev – make in differentiating the *social* from the *societal*. A distinction between the two terms is theoretically necessary – though frequently not made – because the adjective social is insufficient when the actions of two persons are oriented toward a third, society.[3] Thus, something modified by the adjective *societal* is indeed universal (within society), whereas something modified by the adjective *social* does not have to be universal, though it has currency within a specific group.

One important difference between the original framing of the theory of knowledge objectification and the present Spinozist-Marxian approach lies in the different conceptions of how actions are reproduced within a group and, thereby, become part of a cultural repertoire and constitutive of a cultural practice. To illustrate this process, the observations of Japanese researchers how chimps learn to crack nuts is repeatedly used to articulate the fundamental aspects of the *theory of knowledge objectification*.[4] Although the acquisition of the cultural practice is mimetic (i.e., through copying), it does not occur instantly. Accordingly, young

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[1] Vinçent Font, Juan D. Godino, and Jesús Gallardo, 'The Emergence of Objects from Mathematical Practices', *Educational Studies in Mathematics* 82 (2013), 113.

[2] Evald V. Ilyenkov, 'Dialectics of the Ideal', *Historical Materialism 20* no. 2 (2012), 167.

[3] Bringfriede Scheu and Otger Autrata, *Theorie Sozialer Arbeit – Gestaltung des Sozialen als Grundlage* (Wiesbaden: VS Verlag für Sozialwissenschaften | Springer Fachmedien, 2011), 104–107.

[4] See, for example, Luis Radford, 'The Epistemological Foundations of the Theory of Objectification', *Isonomia – Epistemologia* 7 (2015): 127–149.

chimps learn, over the course of three to seven years, how to crack by watching their mothers – with whom the young ones spend the first four to five years of their lives. They watch and try it on their own, and, over time, get to chain the required steps such that they eventually succeed in moving from picking up the nut, placing it on a stone serving as anvil, holding the hammer stone, hitting the nut, and eating the kernel. In this case of knowing to crack nuts, '"knowledge" … is a *culturally codified ensemble of actions*'.[5] Here the adjective 'codified' indicates that the knowledge is cultural, thus general, and, therefore, is not the same as the particular sequence of coordinated actions with some specific stones. Instead, knowledge exists in the form of crystallized labor in general.

This account, however, is problematic, because it can be reduced to biological and, thus, individualistic takes. (See also, *auto*poiesis of the organism in constructivist and enactivist accounts.) Imitation, mimesis, does exist in various forms among animals, and constitutes a condition for higher psychological functions. But the one central to the specifically *human* development, involving 'understanding', exists 'rarely in chimpanzees, and, what is most important, only when a suitable situation and solution of it are within approximately the same boundaries that exist in the chimpanzee'.[6] The account lacks what distinguishes the Spinozist-Marxian and Vygotskian accounts of the development of *higher, quintessentially human, functions*, which inherently are of *societal* in nature being realized *as* societal relations. There is nothing in the preceding account that makes chimp *society* a necessity, for it is from the mother or significant others that young chimp can learn by observing and trying. Vygotsky thus notes that the higher forms of imitation maintain the same trajectory of cultural development as all other psychological functions, which exist *as* (not merely *in*) societal, that is, specifically human relations. Present in Vygotsky's but missing in Radford's account is the role of evaluation integral in the reproduction of practices, which occurs, for example, through corrective instruction or simple declaration of the insufficiency of a preceding (verbal, material) action. Vygotsky points out, referring to the well-known experiments of the gestalt psychologist Köhler, that a chimpanzee 'can meaningfully carry out through imitation only what he can carry out independently' and that 'imitation does not move the chimpanzee further along in the domain of intellectual operations'.[7] Vygotsky argues that the human child is in a very similar situation, but that his *developmental* advance lies in the relation with another human being (e.g., in the cooperation with another human being where a zone of proximal development emerges in joint labor, a teacher's help, whether s/he immediately or invisibly present).

A very different way of learning among bonobo is found in a study that shows the primacy of the relation in the emergence of signing behavior. The particular

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[5] Luis Radford, 'Three Key Concepts of the Theory of Objectification: Knowledge, Knowing, and Learning', *Journal of Research in Mathematics Education* 2 (2013), 12.

[6] Lev S. Vygotsky, *The Collected Works of L. S. Vygotsky, Vol. 4: The History of the Development of Higher Mental Functions* (New York: Springer, 1997), 96.

[7] Lev S. Vygotsky, *The Collected Works of L. S. Vygotsky, Vol. 1: Problems of General Psychology* (New York: Springer, 1987), 210.

case pertains to the development of a symbolic gesture among bonobo chimp that is used by the infant when initiating a pickup.[8] Here, initially the mother picks up a child in a coordinated set of collective movements requiring each of the two parties to move in particular ways to succeed in the pick up that is then followed by the mother carrying the baby. They are doing the pick up *together*, through joint, inherently social labor. Later, the young bonobo will produce part of the movement of the pickup even when the mother is not near enough for an immediate pickup. It is a frozen movement that leads the mother to come closer so that she can pick up the child. Here, the child on its own produces part of what was a collective movement first; that 'frozen movement' signals the intention (desire) to be carried. This act is reified, becomes an act of signaling, as soon as the mother begins picking up the child as soon as the latter produces the frozen movement. Although produced individually, the movement nevertheless is social, having symbolic function that is inherently social.

The difference with the preceding account of cracking nuts is apparent. In the nut-cracking instance, the young chimpanzee looks and then imitates the mother. In the bonobo case, there first is joint labor. Later, when the child reproduces part of this labor on its own, the movement is treated as a sign when the mother actually picks up the infant. That is, a part of the pickup movement now comes to function as a symbol for the pickup event as a whole. That is, the sensible (perceivable) movement on the part of the child now also has a supersensible dimension in that it synecdochically reflects the whole. But that frozen movement does not have this supersensible dimension on its own, nor is the supersensible dimension in the bonobo infant's head. Instead, the supersensible dimension lives in the actual labor of initiating a pickup prior to the actual pickup and with the mother further away from the infant. The genetic origin of that frozen movement is not an 'imitation', is not a 'cultural mimesis', but actual participation in the actual, sensuous pickup event.

Human Essence and Societal Relations

In most philosophical approaches, the essence of human beings, that is, the feature/s that distinguish them from other living beings, is their thinking. The thinking is reduced to mind (brain), which is said to (somehow) have provided an evolutionary advantage to the species. This advantage radically differentiated *Homo sapiens* from other animals during anthropogenesis. What makes humans human was taken as some feature inherent in individuals and thereby naturally making the part of the same class (species) of things. The question of the human essence or the specificity of human nature was a categorization problem where some property is abstracted and all objects having it are classified – similar to defining a class R as

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[8] Edwin Hutchins and Christine M. Johnson, 'Modeling the Emergence of Language as an Embodied Collective Cognitive Activity', *Topics in Cognitive Science* 1 (2009), 534–540.

R = {x : x = red}, where redness is an abstract category. To all the philosophies preceding him, including all materialist philosophers – the latest of which had been his cotemporary Ludwig Feuerbach – Marx responded that 'the essence of man is no abstraction inherent in each single individual. In its reality it is the ensemble of the societal relations'.[9] Philosophers who do not take into account the historical development of humans and society then consider the human essence as something abstract in the sense that constructivist philosopher Immanuel Kant treated abstraction. If, therefore, the particular forms of thinking specific to humans set them apart from other animals, including their nearby relatives such as the various primate genera, then this thinking has to exist *as* their relations specific to human society. Those aspects specific to a society generally are referred to as 'cultural', as in 'cultural objects' and 'cultural practices'.

Marx's position has had considerable consequences for the social psychology of Vygotsky, who, in the fragmentary text entitled 'Concrete Human Psychology', directly refers to the philosopher when articulating the process of personality development: Personality becomes for itself after having existed for other people. From this, Vygotsky concludes that anything internal in higher psychological functions necessarily was external, that is, was a feature of society. That is, 'any higher psychological function was external, this means it was social; before becoming a function, it was a social relation between two people'.[10] Stated in general form, therefore, '*the relation between higher psychological functions was at one time a physical relation between people*'; and, stated in developmental terms, '*the real relation of psychological functions is genetically [developmentally] linked to real relations between people*'.[11]

With respect to method, cultural functions are to be studied through personification of the higher psychological functions. Thus, when investigators focus on the relations between two people – such as a parent and a child or a teacher and a child – then, from the perspective of the parent or teacher, there is a '*renewed division into two of what had been fused in one* … the experimental unfolding of a higher process … into a *small drama*'.[12] That is, for the parent or teacher, some higher psychological function, such as mathematical reasoning, was a social relation first and then became a form of performance produced even when others were not present. When the parent or teacher relates to a child, then what has become one for the adult (i.e. individual behavior) becomes joint labor again (i.e. collective behavior). In other words, from the teacher and parent perspective, what first *was* a form of joint labor became a form of labor that could be performed individually; and now, relating to the child, the individual behavior again becomes a form of distributed, joint labor. It is quite apparent that is unnecessary – and, as Vygotsky writes, even ridiculous – to look into the brain to find centers or regions of higher psychological functions. Instead, readers of 'Concrete Human Psychology' are urged to explain higher functions in external terms and through the study of the drama in

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[9] Karl Marx, and Friedrich Engels, *Werke Band 3* (Berlin: Dietz, 1978), 6.
[10] Vygotsky, 'Concrete Human Psychology', *Soviet Psychology* 27 no. 2 (1989), 56.
[11] Vygotsky, 'Concrete Human Psychology', 57.
[12] Vygotsky, 'Concrete Human Psychology', 58.

and of human relations. Taking a sociological stance, we can reformulate Vygotsky's take, now beginning with the social. Thus, society lives in the different forms of relations of its members. Society reproduces itself by 'acquiring' or 'hiring' new staff that produce the relations that sustain society. What a relation accomplishes, a single person also can accomplish. We may think of that *what* as now existing in the relation of the person to itself as if it were another person, that is, the relation of the self to the other within the self. That relation is of the same form as the social/societal relation with another person.

Doubtful readers may ask: 'Is there any empirical evidence for the position Vygotsky articulates?' The answer is yes, and it is so in a very dramatic way, as shown in the case of the development of deaf-blind children in the Soviet Union, the result of the work of Alexander Meshcheryakov.[13] Commenting on this work, a philosopher of psychology, Felix Mikhailov, would later note that the Meshcheryakov 'directed an outstanding research project and made a notable contribution to the method of practical realization of the truth of Marxist philosophy. The human being, as the subject of conscious, goal-oriented creative activity is formed in intercourse with other people'.[14] The work of Meshcheryakov documents the trajectory from the 'complete absence of mind', to the '*first elementary form* of this mind', and to 'all the riches of the developed human mind, up to its highest and most refined levels'.[15]

The children in the hospital where Meshcheryakov worked were born deaf and blind or lost hearing and sight early in their childhood. They did not act upon things in the world or engaged in exploration by means of touch. When untutored, they may have spent years crouched in the corner of a room or in bed; and they did not generally walk or eat and drink in a typically human fashion. When given a novel object, such as a spoon, they did not explore it but dropped it instead or threw it away. Meshcheryakov organized the children's lives around relations with others, relations in which objects already significant in aspects of society became an organizing means and the medium of *joint* action between the deaf-blind children and their caregivers.

The importance and locus of behavior *as* relation and as joint behavior, was described, among others, for learning to eat with a spoon. Thus, the caregiver would take the child's hand so that they could together search for and pick up the spoon, load it up with some food, and move it to the child's mouth. This joint action, therefore, came to be part of a sequence of movements that had eating and need satisfaction as its ultimate end, its motive. That motive, therefore, was not the result of the child's 'construction' or of some 'innate' intentional capacity or skill. Instead, the motive initially was a result of the relation with another person. Most importantly, the action was not only directed toward the motive, food in the mouth and eating, but also the actual relation with another person. The caregiver even had

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[13] The following summarizes the work described in Alexander Meshcheryakov, *Awakening to Life: On the Education of Deaf-Blind Children in the Soviet Union* (Moscow: Progress Publishers, 1979).

[14] Felix T. Mikhailov, *The Riddle of the Self* (Moscow: Progress Publishers, 1980), 259–260.

[15] Evald V. Il'enkov, 'A Contribution to a Conversation about Meshcheriakov', *Journal of Russian and East European Psychology* 45 no. 4 (2007), 87.

to teach the child to pick up the food from a spoon – using movements of the spoon touching the lips of the children in ways initiating them to pick up the food without spilling it. Before the child had appropriated the motive of the activity, s/he would drop the spoon as soon as the caregiver let its hand go. The caregiver would then initially hold the child's hand and then gradually loosen the grip in the course of which the child learned to keep hold of the tool. Again, the joint behavior – and the social relation it constituted – initially was what subsequently was observed as the behavior of the child. The children also had to learn distinguishing between the activity as a whole and the tool, that is, action with the tool from the tool itself. Only through this separation did the tool become a tool rather than something invisible – much in the way that the wearers of glasses normally are unaware of wearing them and much in the way in which fish are unaware of the waters in which they swim. Again, the joint exploration of the spoon makes it a target (object), which, in the process, comes to be something independent of the feeding/eating activity.

In this extended learning-to-eat-with-a-spoon event, the spoon also became a societal-cultural object, precisely because of its role as the means of societal relation and as the goal of joint action. The object, therefore, did not come to exist somehow by external stimulation of the retina and the brain – i.e., by the neurodynamics of perception. Instead, the child finds the object and, 'by the action of its organs, establishes the object's attributes as existing outside itself, repeating, as it were, reproducing them by its movements only because it has been drawn into intercourse, the mode of which separates action with the object from the object itself'.[16] That is, it is in the relation the children not only learn to eat with the spoon but also *about* their relation to the food and eating by means of the spoon. Their relation becomes a relation for a second time; and this 'doubling' of a relation is characteristic of human beings. This new relation comes to be associated with objects and words, becoming the supersensible parts that go with the sensible body of things and sounds. The counterpart of the supersensible part of the objects is the ideal ('meaning') in the mind of the individual. As Marx writes, 'the animal does not 'relate' itself to anything, it does not 'relate itself at all. For the animal its relation to others does not exist as a relation'.[17] Thus, for the human beings, consciousness is a social product from the beginning, in this special case, the facts of eating and eating with a spoon both become aspects of consciousness in and through the relation: they are material social relations first before they become aspects of consciousness.

We may summarize the above by stating that the work of Meshcheryakov exhibits how '*the modes* of [the human being] develop historically and *the means* of which reserve in themselves the universal ([societal]) determinates of all the objects of his activity'.[18] Important in this quotation is the fact that 'the universal' is associated with 'the societal' rather than 'the social'. This points us to a way of

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[16] Mikhailov, *Riddle*, 261–262.
[17] Marx and Engels, *Werke Band 3*, 30.
[18] Mikhailov, *Riddle,* 260.

distinguishing the forms of relations between students that are social but not 'societal' and therefore not universal. It is precisely relations involving the teacher, who already exhibits universal forms of behavior (e.g. mathematical reasoning) that 'cultural' accomplishments, universal forms of mathematics, more likely come into existence than in relations between students.

Sociogenesis of Mathematical Reasoning

In the classical mathematics education literature, especially that part that affiliates with social-constructivist and sociocultural theories, there is a lot of writing about how students or students and teachers together somehow 'socially' 'construct' mathematics. Because there are groups involved, the construction is said to be a social one. Once some aspect of mathematics is constructed socially, so the ideological discourse goes, it can be internalized or individually (re-) constructed by the student. There are claims that this idea was disseminated with the writings of Vygotsky. In the preceding section, evidence is provided that Vygotsky wrote something very different. The text he wrote asserts that any higher psychological function *was* a social relation first. That is, the transactional order of the give and take that can be observed when students engage in the joint labor of doing mathematics – with or without teacher – eventually is the same order that the individual exhibits. That is, all the while acknowledging that something occurs in the relation, Vygotsky emphasized that at some point in time, the societal relation itself *is* the first time that the child participates in a particular form of collective, *universal* behavior that subsequently will show up as his own behavior. The way in which the statement is framed means that not all soci(et)al relations will be psychological functions and relations thereof, but all psychological functions and relations *were* relations with others first. For example, in the context of reading, orienting to features of the page, or establishing the particular movement from the end of one line of text to the beginning of another line, have been shown to be relations in the very early life of (middle class) children that may exist even before a child actually articulates the first word.[19] Vygotsky himself uses the example of the development of intentional pointing, which first is a movement that the mother subsequently treats as pointing by giving the child the object that lies in the extension of the hand movement, and finally the child uses the movement to point. In the following, I present a case from early mathematics education using materials collected in the same second-grade class in which Melissa, Sylvia, and Jane featured in chapter 2 were students.

   The particular case presented here relates to mathematical reasoning. As a specifically human characteristic, mathematical reasoning *is* a social relation first (if Vygotsky and Marx are right). The particular form of mathematical reasoning is

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[19] Wolff-Michael Roth and Alfredo Jornet, *Understanding Educational Psychology: A Late Vygotskian, Spinozist Approach* (Dordrecht: Springer, 2017).
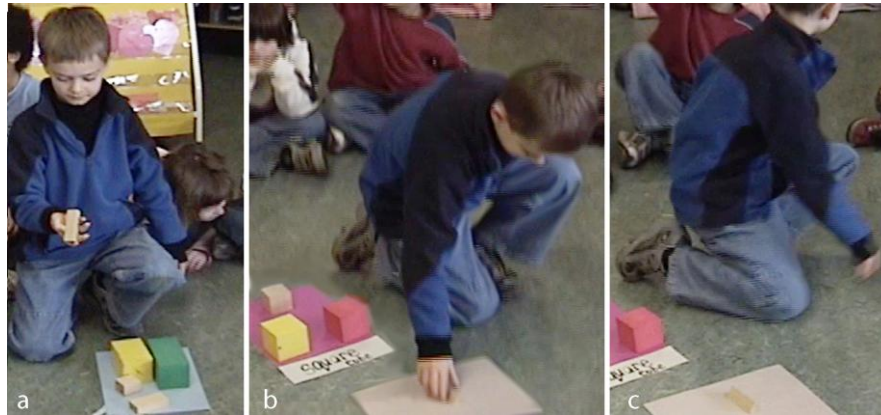
Fig. 4.1 Chris gazes at his object while turning it about several times, then places it on a mat of its own next to the cubes, and, without saying anything, heads toward his place in the circle

that of classifying an object followed by a verbal account thereof (i.e. a 'reason'). The classificatory action may be performed by (a) placing an object with an existing collection of objects or by beginning a new collection or (b) making a statement such as 'this one [object] is a [type of object]', and then providing a reason for the placement or statement. The action then can be written as 'doing [classifying by means of a geometrical property]'. In this class, geometry was introduced during the first lesson by means of a classification task that the whole class engaged together. The children were assembling into a circle an then, one at a time, after taking a mystery object from a black plastic bag, either placed it with an existing group of objects in the increasing classification system or created a new group by placing the object on an empty mat (see Fig. 4.1). During this lesson, only one of the 22 students spontaneously provided a reason. In the case of all other students, the event unfolded as described in the following example.

Chris reaches into the black plastic bag and pulls an object. He gazes at it, turning it repeatedly (Fig. 4.1a), then moves forward placing the object on an empty mat incidentally situated right next to the mat with the label 'cubes or squares' (Fig. 4.1b), and finally retreats to his seat (Fig. 4.1c). Even though there already have been 14 classifications, plus the one that the teacher had used to start up the task, in each case of which the teacher repeated that they have to state their reason, Chris does not explain his thinking (provide a reason). But when such a statement is requested, Chris does provide it, as shown in Fragment 4.1, which begins with the invitation and some organizational talk. After what comes to be the completion of the turn and a pause, a first reply is offered up (turn 03), but it, too, does not suffice in the face of a subsequent one-word turn 'because' with increasing pitch such that it may be heard as a question. It is a kind of turn that, because unfinished, does in fact offer up a slot to be filled, a designedly incomplete turn – here instantiated by the singular 'because' – so that something can be provided to complete it. In this, the turn also functions as an evaluation of the preceding one, marking it as
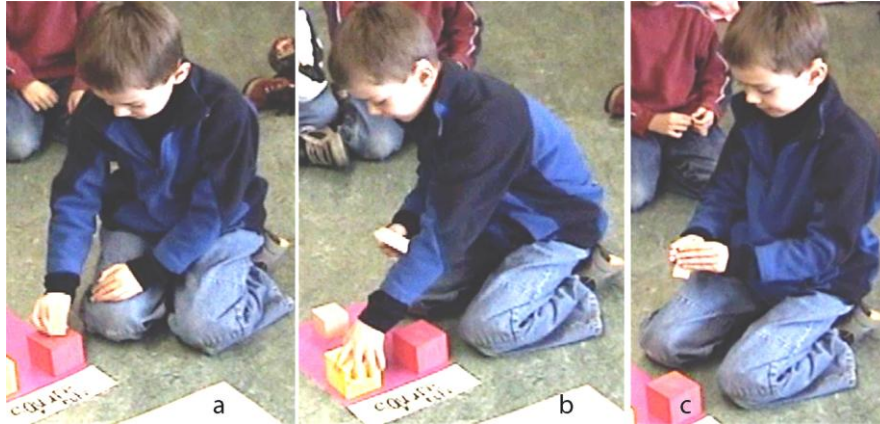
Fig. 4.2 Chris, once invited, provides a reason for the preceding classificatory action by holding the prism against a cube (a), then touching faces of the cube (b), and finally folding his hands around the prism (c)

incomplete. It is the intonation that allows hearing this turn as a question. The incompleteness requests completion and the conjunctive 'because' offers up a link that would connect what preceded – i.e. the statement in turn 03 – to that which is projected to follow – something of the type of the statement that actually follows (turn 06).

Fragment 4.1
01 T: before we go on chris has to explain his thinking and during this time I want to remind you braden (.) that you need to be quiet so chris gets his turn to explain his thinking. hands down! ((*Chris has come back to the groupings and has picked up his object*.))
02   (0.9)
03 C: [it definitely doesnt go with   ] this one
       [((*holding his prism against a cube, Fig. 4.2a*))]
04 T: be:c*au::*se?
05   (0.5)
06 C: um (1.1) the (0.6) these ones have more (0.2) sides ((*points to two orthogonal faces, e.g., Fig. 4.2a*)) (0.2) and >this one only has< (0.3) three sides ((*hand configuration as in Fig. 4.2b*)).

The phrase offered up in reply to the invitation to articulate thinking comes to be treated as insufficient, just as the physical act of classification before. However, through the joint work of this stretch of classroom talk, the classificatory actions – first in the form of a physical placement, then as a verbal statement ('does not go with this one') – comes to be connected to a reason. Whatever the thinking that occurred in the course of placing and stating before, it now comes to be articulated in terms of the number of sides that distinguish the prism from the cube. The work

of making this connection is social through and through because, for reasons of the analyses provided in chapter 3, each turn belongs to both, constituting the labor as common rather than as sum of individual work. In fact, it is not just social but indeed universally recognized as mathematical and, therefore, deserves the adjective societal: We see in this the *mathematics* of mathematics, and our capacity to see the mathematical in these actions is itself evidence of its universality.

Several points are important in this episode. First, Chris already produces the first and second part of the action, the doing of a classification and a corresponding (here verbal) account. He therefore does not have to internalize something because – were we actually desiring to uphold an emphasis on the internal–external dichotomy – it would already have been internal to that thinking body. Second, readers familiar with the literature on *scaffolding* – an idea sometimes explicitly attributed to Vygotsky – might point out that it is the teacher who somehow makes available the resources, here in the form of the connective 'because', that allow Chris to combine the two parts of an action. But, for the reasons articulated in chapter 3, 'because' does not just belong to the teacher. If it has any effect on the unfolding talk and task, it also is a word that belongs to Chris – to repeat: Vygotsky says that the word is impossible for one but a reality for two persons. Third, then, even though Chris already has all the parts, the whole *mathematical* act of classification, he experiences it for a first time here in the joint labor with his teacher.

In the course of the three-week unit, most of the children – at different points in the curriculum trajectory – came to the point where they performed on their own what during the first lesson was distributed, joint labor that also included the teacher. But even though the children began to reason mathematically without teacher intervention – e.g. when talking with peers in the completion of tasks – it was no less societal (i.e. universal). It would be no less societal if they had been working on their own and writing down the reason. It is social so that it can be unfolded again into a small bit of drama, exemplified here in the persona of the teacher. Thus, at some point in her life, the teacher was in the same kind of situation as the children in her care. Reasoning mathematically first was for her a societal relation with another, then it became a behavior that she displayed even without other people present – e.g. while taking university mathematics courses. In the present lesson, what was fused into one, to speak with Vygotsky, was again divided into two, into the social drama of a real-life classroom. There is lots of evidence to show that for the children here what has been a little social drama – evaluation of an action as insufficient, invitation to producing a completion, and eventual evaluation that completion was satisfactorily achieved (by moving on to the next student, which only occurred after the previous classification was successful).

To stay with Chris, we have one such episode (and there are many for him that could have been chosen) where he exhibited the fused behavior. It occurred one week after the preceding classroom fragment. In one part of that subsequent lesson, the teacher involved the children in talk about the different mathematical properties of three-dimensional objects – number of edges, vertices, faces, the shape of the faces, etc. She made reference to a poster that included drawings of objects to which key terms were linked. At one point, what can be heard as an invitation to

Fig. 4.3 Chris reasons mathematically by stating a classificatory proposition and then providing a reason

make descriptive statements about the faces of geometrical objects, there is an accepted (turn 03) request (turn 02) for showing something. Chris takes his time walking up to the front. He picks up a pizza box situated just below the chalkboard and poster, and then provides what can be heard as a proposition immediately followed by an explanation that has verbal and gestural parts (Fig. 4.3). It classifies the object as a rectangular prism, and then provides a reason, which exists in the different lengths of two orthogonal sides. (We are at present not concerned with the fact that a pizza box probably is square, which is consistent with what Chris will state next when he says that the pizza box is like a 'flatt[ened] cube'.)

Fragment 4.2
01 T: what else about faces can we can we describe ((*looks around*)) chris?
02 C: can I show you this?
03 T: um okay
04   (4.3) ((*Chris walks up to the front*))
05 C: um
           (0.9) ((*arrives at front of classroom next to the teacher*))
           I think this one's kind of ((*picks up pizza box*)) more like a rectangular
           prism because this one ((*moves long one side, Fig. 4.3a*)) like (0.4) is
           longer than ((*touches other side of box, Fig. 4.3b, moves twice along*))
           this w (0.2) than this one is.

In this instance, the connective introducing a reason is followed by a qualitative comparison of the measures of two orthogonal sides of the same object of the type '*x* longer than *y*' or '*x* > *y*'. It is an instance where quantification is used. This is different from just using shape, which characterized much of the first lesson – though in that first lesson it was precisely Chris who already made a distinction

based on the number of sides between cubes and the new kind of object he had drawn from the bag, the prism.

Materiality of Relations with Others

It may be because of the intellectual orientation of existing theories of learning and development that the materiality of human relations is forgotten, a materiality that exists even if the people involved are not actually touching each other. Those who profess at least partially an intellectual heritage from the ideas that can be found in the writings of Vygotsky appear to attend little if at all to the fact that those relations in which a higher psychological function first appears is in fact a material relation: '*the relation between higher psychological functions was at one time a physical relation between people*'.[20] Acting on others, an important aspect in the sociogenesis of thinking and reasoning, does not occur through some form of 'ether', but precisely takes physical form. This is so not in the least because all of our senses are based on materiality: photons impinging on the retina, 'moving layers of air' entraining the movement of the ear drum and malleus, the chemical substances entailing processes on our tongues, or the worldly objects that stimulate the sense of touch. It is not in the least the materiality of the *sound*-word that is so important to Vygotsky because it is the sound part required for the affecting and affective nature of human relations. It is the sound part that allows us to hear a voice – and, by attribution, the person producing the sound – as sensitive or insensitive, aggressive or empathetic, as patient or rushing. It is that same materiality that also plays a role, though regarding the voice insufficiently elaborated, in the theory of knowledge objectification. There are many situations where the relation tends to involve physical contact, which might occur, for example, in learning how to hold a pencil or pen, how to use a (construction) tool, and so on. Although many experiences involving physical contact might occur at home with parents, we should not assume that children actually have had the experiences that are required for doing mathematics and thinking mathematically.

   In this part of the lesson, Mrs. T. and Mrs. W. are conducting the lesson together. Mrs. T. has asked children about edges and what they could say about them. Thomas has raised his hand, is called upon, and, bringing with him the cube from the desk, then comes to the front of the classroom where a poster with the parts of three-dimensional objects was fastened to the chalkboard. In the exchange, the term 'vertices' comes to be treated as that object to which the Thomas' (t) hand-arm movement and index finger are oriented.[21] There is an apparent invitation (verbal) to place the finger on the vertex together with the movement of the index

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[20] Vygotsky, 'Concrete Human Psychology', 56.

[21] There is in fact a mutually constitutive relation at work, where the object is required to determine the motive of the hand-arm movement and index finger, and the latter are required to determine the former as the object intended.

finger that terminates on the vertex (Fig. 4.4a); but the invited finger moves to the topmost face of the cube (Fig. 4.4b). While Mrs. W. is 'capturing' Thomas' finger, a statement unfolds in which the finger is described as pointing toward one vertex (turn 21b), all the while her hand moves Thomas' index it holds until it touches the vertex (Fig. 4.4d). There is an invitation to run the finger along 'that' 'edge', and, as the words unfold, Mrs. W. moves Thomas' index finger along the edge (turn 21c, Fig. 4.4e). There is a longer pause, and an interrogative, followed by another pause during which the movement along the edge is repeated. There is then an of-fer of an invitation to describe what happened, followed by an offered invitation to state the kind of edge they are dealing with, while the joint movement of Thomas' index in the hand of Mrs. W. along the edge is repeated three times. Thomas does not speak. But in the seemingly eternally long pause that unfolds, Thomas first moves his index along the edge on his own, then lowers his head, which he scratches (Fig. 4.4g). In the course of the unfolding lesson, it comes to be treated as what teachers often call a 'teachable moment', an instant that appears to call for teaching because the student has manifested not knowing or not understanding.

Fragment 4.3
21   W: but but its p*art* of *on*e verticee; put your f*in*ger on th*at* ((*Fig. 4.4a*)) verti-
          cee.

21a   [(0.82)
     t: [((*Fig. 4.4b*))]

  b W: your f*in*ger ((*Fig. 4.4c*)) (0.26) [p*oi*nt (0.62) *one* verticee]
                                              [((*Fig 4.4d*))   ]

  c   (0.48) then >r*un*:it=[along that edge< to] the *oth*er verticee
                               [((*Fig. 4.4e*))   ]

  d   (0.77)
  e   °okay?°
  f   [(1.24)   ]
          [((*Repeated movement along edge*))]

g   [whats that f*eel* like; >what kind of *edg*e is that.<]
          [((*3x repeated movement along edge*))   ]

22a   [(4.22)   ]
     t: [((*moves finger along edge twice, Fig. 4.4f;*]
          [*finally scratches his head, Fig. 4.4g*))   ]

    In this lesson fragment, we observe a number of things that are important in and to psychological development. Three points are to be made: (a) the worldliness (materiality) of words, (b) the doubling of relations and experiences, and (c) the *real* (physical) relation between two people.

Fig. 4.4 In this Lesson fragment, Mr. W. takes Thomas by his hand, and in their joint movement, he gets to feel the edge

Worldly Words

First, we observe Mrs. W. take Thomas' index finger, place it against a feature of the object naming it 'vertex', and then move the finger along what she names 'that edge'. Her hand movement takes Thomas' index finger on a tactile exploration. The movement is theirs, though there also is a division of labor, whereby one movement entrains the other more passive one (though still active in that it lets itself be moved). There is therefore an opportunity for an experience in which (a) a feature of the physical world, culturally referred to as 'edge', can be felt by means of the sense of touch and (b) a specific sound-word is articulated at the same time. Readers familiar with Ludwig Wittgenstein's *Philosophical Investigations* immediately may be reminded of the similarity between the present fragment and what a pragmatic approach to language classifies as the teaching of a *language-game* – defined as the whole of language and the actions with which it is intermeshed – where 'children are brought up to perform *these* actions, to use *these* words as they do so, and to react in *this* way to the words of others. Words, as material entities, are just as much part of the physical environment as all the other things including other human bodies that constitute the surroundings. An important part of the training will consist in the teacher's pointing to the object, directing the child's attention to them, and at the same time uttering a word'.[22] In the present fragment, the teacher is not just pointing to, but guiding the student's hand in a movement along the edge, which becomes an instant of '"ostensive teaching of words"', which 'can be said to establish an association between the word and the thing'.[23] The issue is, of course, not a simple one because the *what* of the ostension is itself underdetermined, which can be seen in the absence of a verbal reply following an apparent invitation to describe the edge and in a subsequent index finger movement through the center of the face when invited to indicate an edge (see chapter 8). On the other hand, touching and moving are resources for decreasing the possible features that 'edge' is to be associated with. As Fig. 1.1 shows, there is more than one gestalt that may clearly appear against the ground given the same configuration. With which of the possible features is the word associated? If there is a figure, clearly it offers itself as the one intended.

   Important to retain is the fact that at issue here is more than a thing, because apparent to us – but not for Thomas – Mrs. W. is seeking to guide the student to attend to the edge rather than some other feature (e.g. the vertex he had oriented others to) of his object.

   The type of movement we observe in the fragment has been referred to as *epistemic*, in contrast to *ergotic* or *symbolic* movements that may take exactly the same form. Both the sense-based epistemic movements exploring and learning about the world and the (work-related) ergotic movements may evolve into symbolic movements. That is, initially these movements are part of finding out shapes and tex-

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[22] Ludwig Wittgenstein, *Philosophische Untersuchungen: Zweite Auflage / Philosophical Investigations: Second Edition* (Oxford: Blackwell, 1997), 5.

[23] Wittgenstein, *Untersuchungen*, 4.

tures or changing some physical configuration and later serve communicative purposes in descriptions and explanations.[24] Thus, for example, in chapter 2, the same finger configuration that functions as a measure or referent in determining the relative lengths of the edges of a cube also serves to communicate what was done and what has to be done to determine whether an object is a cube.

Another important point related to this first issue is the fact that there is more than a finger running along the edge. Simultaneously, there is an opportunity for the eyes to follow. When the eyes follow the movement of the finger, then the very type of movement required for seeing a straight line is performed. The phenomenon is not unlike that observed in learning to read, where the parent or institutional teacher may take the finger of the child and move it just beneath the words (to be) read. When the end of the line is reached, the finger is moved to the beginning of the 'next line'. In this way, the finger movement entrains the movement of the eyes to follow the words and find the continuation to the next words on the line below.[25] Once a child has taught its eye to make this movement, the finger no longer is required to guide the eyes along – though in some instances, even college students might find themselves having to use their index finger in this way. Indeed, we observe the research scientists and PhD students in chapter 3 do precisely that when they use their pencil tips to follow certain graphical features and, in this movement, allow these to become figure (gestalt) above the athematic ground. The difference between the scientists and Thomas is that they do so without being invited, though in other contexts, one scientist may explicitly point to and move along features, which then entrains the eyes of others to perceive the relevant gestalt.

Doubling Relations and Experience

Second, an important feature of the fragment is the fact that Thomas is invited to do more than just feel vertices and edges. Thomas is invited to state what *it* – where the nature of it remains unspecified, though it may be taken to refer to the movement along the edge – 'feels like'. There is not just an instance of physically touching the vertices and moving along the edge. Instead, there is an invitation for doubling the experience by providing a verbal account. That is, there is an invitation to make the relation between finger and object – touching or moving along – a relation. It is a move to bring the physical properties of an object, vertices and edges, into conscious awareness. This then means that there is a material relation, which is doubled in associated talk. This, as pointed out in chapter 1, is the core point in the Spinozist-Marxian conception of human relations as distinct from the ways in which other animals relate to the world – likely with some graded transi-

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[24] See, for example, Wolff-Michael Roth, 'The Emergence of Signs in Hands-on Science', in *International Handbook of Semiotics,* ed. Peter Trifonas (Dordrecht, The Netherlands: Springer, 2015), 1271–1289.

[25] Roth and Jornet, *Educational Psychology*, chapter 7.

tions. Readers recall that Marx and Engels write: 'where a relation exists it exists for me, the animal does not "*relate*" itself to anything, does not "*relate*" itself at all. For the animal its relation to others does not exist as a relation'.[26] Here – as in the case of the deaf-blind children learning to eat with a spoon, where the awareness of the spoon as tool between food in the plate and food in the mouth is required for a human-type relation between food and eating – in addition to touching and moving, the feeling of movements within the body are to be articulated in symbolic form.

In many classrooms, the verbal part of such events has primacy over other parts. In fact, it is the supersensible aspect of the words that has primacy, the 'meanings' that in the constructivist tradition are ideal things, inherently inaccessible as formulated in Plato's allegory of the cave dwellers who only see shadows but never the light that casts them. This is so even when opportunities are to be created and provided for having students engage with materials or observe certain events. The problem that can be observed in many instructional sequences is that the teachers fail to establish what the students actually feel, hear, see, touch, or smell. But those senses are important because the senses of the body constitute the body of the sense they develop. An assumption is made that students see whatever teachers intend students to see. But this is a fallacy. Thus, one research project in a physics classroom clearly exhibited the fact that the students saw very different things in the same demonstration.[27] Under experimental conditions, the same event was seen as involving movement ($n = 18$ students) and as involving no (or to be neglected) movement ($n = 5$ students). The explanations also differed substantially, for the majority drew on some physics discourse to explain why no movement was observed, whereas the remaining students explained why there should be no movement. During the original lecture, which involved a structurally equivalent demonstration, the teacher had simply continued assuming that the students all had seen the phenomenon in the way that the teacher saw it and in the way that the lecture subsequently explained.

Real Relation Between Two People

Third, there is more to the event than simply Mrs. W. taking the finger of Thomas to hold it against a vertex and to move it along an edge. In that taking, holding, and moving the finger, there is a real (physical) relation between two people. There is also a physical relation with talk. But in the literature, talk tends to be taken for its semantic aspects, not for its material aspects. They are literally doing the moving along the edge together – Thomas quite apparently allows having his index finger

[26] Marx and Engels, *Werke Band 3*, 30.
[27] Wolff-Michael Roth, Campbell McRobbie, Keith B. Lucas, and Sylvie Boutonné, 'Why Do Students Fail to Learn from Demonstrations? A Social Practice Perspective on Learning in P'hysics', *Journal of Research in Science Teaching* 34: 509–533.

held against the vertex and moved along the edge. The division of labor should not lead us to assume that there is not joint labor occurring. Vygotsky's suggestion to look at social relations as drama helps, because a dramatic event involves all participants, in which ever role they contribute; but contribute they do, even though it may be as patients to which things happen, so that any drama unfolds in and as the joint material actions of the participants.

Important for developmental purposes is that the real, physical relation itself comes to be doubled. It is in this way that the sensible comes to be associated with the supersensible, the ideal. That supersensible also is real because it exists *as* real, physical, and social relation between two people. This is so because what is at some point a higher psychological function was a real (physical) relation first, but the higher function, as a specifically human thing, is a relation only when it exists a second time in consciousness. It is well known in the social sciences that human relations themselves are made the topic of talk by means of formulating practice. There are multiple examples of formulating in the larger teaching | learning event from which the fragment is drawn. Because this larger episode features extensively in chapter 8, it shall suffice noting here to refer to but one of these. After being asked about the difference between the edge on a cylinder and the edge on a cube, Thomas, in the second case, moves the index finger along the four edges of the upward-facing square. When Mrs. W. says, 'if you did just one of them', she simultaneously articulates that Thomas has 'done' more than 'just one of them'. In this, he has not followed the invitation and not given what he was invited to give. The statement evaluates the preceding {invitation | reply}, an aspect of their irreducible joint labor, and makes stand out the misfit between the two parts. Even more salient is this doubling of the relation when there is some form of trouble, such as when, in a fourth-grade mathematics lesson, we overhear the turn pair {'I don't understand though' | 'you don't understand that its what I'm trying to help you understand'}.[28] The second part formulates that whatever preceded the first part, which is evaluated in that part, was (an attempt at) 'trying to help understand'. More generally, therefore, *formulating* is a way of doubling relational experiences, whereby relations *become* relations leading to experiences of experience that are the foundation of anything that is particularly human.

Here, then, two types of relation are visible: between humans and the material world, associated with a parallel relation between human beings. Both types of relation are inherently historical, because the world that a child encountered takes its shape in the context of particular societal practices, which are different today than they were 2,000 years ago, during the height of the Greek and Roman cultures, 5,000 to 6,000 and more years ago during what today is known as the stone age. Relations also change – clearly apparent when there are clashes, such as in the French Revolution, 'gender wars', and other periods that lead to changing societal relations. As Marx and Engels note, 'consciousness is at first, of course, merely consciousness concerning the *immediate* sensuous environment and consciousness

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[28] Wolff-Michael Roth and Luis Radford, *A Cultural-Historical Perspective on Mathematics Teaching and Learning* (Rotterdam: Sense Publishers, 2011), 62–63.

of the limited connection with other persons and things outside the individual who is growing self-conscious … this particular behavior toward nature is determined by the form of society and vice versa'.[29] As a result, human relations with nature are determined by their social relations, and social relations are determined by the relations to nature. The relevance of these points to the present case lies in the fact that opportunities for learning and development arise for Thomas not merely because he is somehow guided through an experience to feel an object and its properties but that this relation to the cube is paralleled by and associated with a second physical-social relation to another person. That relation then, as supersensible, comes to be objectified in and by the material-sensible part, on the one hand, and in the subjective praxis of the person, on the other hand.

Sociogenesis and Drama

In this chapter, I describe and exemplify the phenomenon of sociogenesis, which essentially names the process whereby what eventually will be individual behavior and associated psychological functions, is observable as social-physical relation between people. Relevant to the role of the teacher is the fact that an apparent individual form of behavior is again divided into two, that is, becomes a drama played out on the public stage of the classroom. As seen in chapter 8, Thomas does indeed also glance in the direction of the present audience, as if he were looking for its appreciation of what is happening generally and his not responding in turn 22 specifically (Fig. 4.5).

   In his fragmentary text entitled 'Concrete Human Psychology', where Vygotsky articulates the materially objective relations as the locus of specifically human behavior and higher psychological functions, he also refers to drama, making note specifically of the idea of '*psychology in terms of drama*', because 'the dynamics of the personality = drama, *sociogenesis is the one true perspective, i.e., mechanisms are created in the environment*'[30]; he explicitly cites the French Marxist psychologist Georges Politzer, who had published an article and a book on concrete psychology in which the category of drama played a central role. Politzer emphasizes that we feel surrounded by real persons in their flesh and blood rather than by psychophysical structures, which can be obtained only by abstracting from real people. Together with other persons, we participate in, and produce, human life; and *human life* always is lived out in dramatic form. The term drama as it constitutes everyday human life pertains to all of its aspects, such as interviewing a candidate for a faculty position, playing a game of street hockey, inviting | accepting/-rejecting to go for a cup of coffee, or a debating a case during a tenure-and-

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[29] Marx and Engels, *Werke Band 3*, 31.
[30] Vygotsky, 'Concrete Human Psychology', 58 & 68. See Georges Politzer, *Critique des fondements de la psychologie: La psychologie et la psychanalyse* (Paris: Presses Universitaires de France, 1928); and Georges Politzer, 'Les fondements de la psychologie: Psychologie mythologique et psychologie scientifique', *La Revue de la Psychologie Concrète* 1 (1929): 9–64.

Fig. 4.5   Thomas takes a glance at the audience all the while Mrs. W. has taken his finger, and together they will 'run it along the edge'

promotion meeting. We know one another only in and through our joint participation in (dramatic) events; and so our knowledge itself has to be, and is, dramatic. The comprehension human beings mutually have of each other, for Politzer, is essentially dramatic.

Psychology becomes humanized precisely when it focuses on the drama of social relations, which will become the drama of personality when social behaviors have been fused to become individual behaviors. Important in this conception of drama is not the form it had in the past – where roles were fixed, the idea underlying older forms of psychology – but a form of drama where there is variation in and of roles. Instead of focusing on processes, concrete human psychology is to focus on the whole person involved in the drama of life, consisting of the ensemble of social dramas. Oftentimes we are caught up in events, focusing on what is salient rather than making thematic in consciousness the event us a whole. But at times we become aware that we are in an event, sometimes on a stage for everyone to see. We may then take glances at the audience or bystanders to get a glimpse of the appreciations they make available. This is precisely what can be observed in the case of Thomas, who, repeatedly, glances at the audience even though the play currently focuses on something characterized as mathematical (Fig. 4.5).

In the appreciation of some scholars, drama is an important term in Vygotsky's thinking, though he has not elaborated it much.[31] It is in the dramatic collision between the child's current forms of behavior and the demands from others – who are integral part of the pereživanie [experience]-constituting environment – that we

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[31] Nikolai Veresov, 'Zone of Proximal Development (ZPD): The Hidden Dimension?' In *Språk som kultur – betydningar i tid och rum,* edited by Anna-Lena Østern and Ria Heilä-Ylikallio (Vasa: Åbo Akademi, 2004), 13–30.

find the moving force of development. In our episode featuring Thomas, movement and development can occur precisely because of the contradiction between his orientation toward the vertices and the teacher orientation toward the straightness of the edges on his cube. It is in this event and through their joint labor that the contradiction is to work itself out. Contradiction means difference, and difference over time, change, is brought about by movement. Being-alive means moving and changing, or otherwise we would be talking about something dead. Drama, movement, and contradictions all are but different terms relating to the same event, the once occurring never-repeating drama of Life. The drama itself is created by the event, which, in Vygotsky's the Russian language, especially in respect to film and theater, constitutes a category. In materialist dialectical theorizing, a category of thought also defines a unit of analysis – including 'word-signification [značenie slova]' or 'perezivanie [experience]' in Vygotsky's work or, in cultural-historical activity theory, 'activity', 'consciousness', and 'personality'.

Drama also involves affect. In fact, the whole purpose of the dramatic genre is to emphasize the role of affect and emotions in every part of our lives. Although we tend to perceive manifestations of affect and emotion in whatever event we participate – the calmness of the department head, the shortness or rudeness of the cashier at the checkout, and the care or overbearance of a parent – it is much harder to find 'objective' evidence that is less amenable to 'mere' interpretation. Nevertheless, we can hear care in the voice of Mrs. W., we see her bend over to assist Thomas in an apparent attempt to help him get into the know about edges. We also see and hear the sometimes-hesitating replies of Thomas, and something like dejection or discouragement, perhaps even mixed with some frustration and shame, during the long pause after he has been invited to state (a) what he was feeling while his finger moved along the edge of the cube and (b) what kind of edge it was (Fig. 4.6). A drama is unfolding here for all those who constitute the audience: the other students, Mrs. T., and the two researchers videotaping the event. This audience can see how the drama continues, when Mrs. W. fetches a cylinder and lets Thomas feel its edge by taking his index finger on the same kind of journey as before. The audience now may understand what happened before in its relation to what they see, which we may gloss in this way: 'Thomas did not know, and now the teacher is helping him learn about edges and then provide the answer that even later events reveal as the one that was sought for all along'.

Vygotsky's work is often used to legitimize the distinction between outside (intersubjective, interpersonal) and inside (intrasubjective, intrapersonal). Thus, students are said to 'construct' with others – peers or teachers – some 'meaning', knowledge, or norm in the social arena and then internalize whatever has been constructed to make it their own. Sociogenesis, once theorized in terms of the category of the dramatic event, affords thinking in a different way, more appropriate to the Spinozist-Marxian take on social psychology that Vygotsky was working toward. Thus, he notes that 'every function in the cultural development of the child appears *on the stage* [*scena* = 'scene'] twice, *in two planes*, first, the social, then the psychological, first between people as an [interpsychological] category, then

Fig. 4.6 Mrs. W. bends over, getting closer to Thomas in her orientation toward helping him produce the requested reply. The long, 4.6-second pause and the change in his bodily composure and orientation are treated in what follows as manifestations of the need for additional help

within the child as a intrapsychological category'.[32] There are two parts to stage, front stage and backstage. These are two integral and irreducible parts to *one and the same* scene, which means, when used in a Vygotskian take, there are simply two dimensions of a *single* developmental event.[33] Thinking about and theorizing the learning and development that Thomas is and will be undergoing, therefore, is not some 'social construction' followed by an 'individual construction' of knowledge. Instead, there is a *single* developmental event that cannot be reduced to the individual or social-collective. In sociogenesis, 'individual' and 'social' merely are two ways of characterizing the same dramatic event, much in the same way that 'duck' and 'rabbit' are two ways of characterizing the same line drawing in chapter 1 (Fig. 1.1). As soon as we theorize educational phenomena in this manner, it no longer makes sense to make a categorical division between inside and outside, Self and Other, intrasubjectivity and intersubjectivity, which is the topic of chapter 5.

An important opportunity arises from this for mathematics education research, because the category and discourse of drama affords making thematic the whole person. There is not just some hidden 'mental framework' involved but the real Thomas and Mrs. W. in their flesh and blood, laboring with and for the other and the subject matter. They participate in a dramatic event, on the same stage, and make available everything required to pull off the ongoing play. That play has its own dynamic, which is not the composite of individual dynamics but an irreducible social dynamic where individuals play their parts that can be understood only as parts of a whole; and when the whole changes, the parts also change. In other words, we may think of Thomas and Mrs. W. as the staff that bring a social phenomenon to life. Because they are part of a social phenomenon, we cannot understand their respective participation independent from the participation of the other and independent from everything else what is part of the dramatic event as a

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[32] Vygotsky, *Collected Works Vol. 4*, 106, emphases added. The inappropriate translations, which render the adjective 'psixičeskij' (on p. 145 of the Russian original) as 'mental' rather than 'psychic', which is used in all translations to languages other than English, or 'psychological', the translation that Aleksandr Luria suggested to Mike Cole.

[33] See Veresov, 'Zone of Proximal Development'.

whole. As a result, what and who children become is the result not of some individual (mental) construction but is the result of their participation in the actual production of classroom life, in its drama: part of the drama of life. If we do not understand the drama of life in which the individual children are involved, we do not understand the changes and developments of needs, interests, inclinations and feelings occur when they participate in the practical material life of society. For students, this participation occurs partially in schools. These issues are further elaborated in chapter 8 concerned with *obučenie* [teaching | learning], a way of bringing into focus all the participants in a situation and the changes they undergo through their involvement in the unfolding, unfinalized event viewed in terms of a developing drama.

In all of this, Vygotsky radically differs from Piaget and all forms of modern constructivists, who begin with the individual self that is socialized as its behaviors are reconfigured to make them appropriate for participation in collective affairs. In Piaget's approach, the child's own ways of thinking are 'domesticated' and its behavior is shaped to conform to the culturally accepted ways of acting. A typical context in which this approach manifests itself is whenever the concept of 'negotiation' is invoked. That very term presupposes that participating individuals are in possession of things that come to be adapted in the process until some form of agreement is achieved. Vygotsky recognizes the characterization of development in the constructivist tradition, which 'assumed that a function exists in the individual in a finished, half-finished, or embryonic form, and that it is exercised, developed, grows more complicated, is enhanced, enriched, inhibited, suppressed, etc., in the collective'.[34] In his Marxian social psychology, on the other hand, development unfolds in a process whereby social functions come to be individualized, and joint behaviors that exist as and institute social relations eventually are individual behaviors. As a result, development proceeds toward *individualization* of social functions rather than toward socialization; there therefore is a movement from and transformation of social functions into psychological functions. The question then is one of how the collective 'captures' another child, so to speak, turning it into a staff that keeps the collective phenomenon alive.

But there is more to consider because the social relations are not constant. Readers know this intuitively based on the fact that parents and institutional teachers change while parenting and teaching. This means that physical-social relations change not only children but also the significant adults in the children's lives. They simultaneously change social practices, and changing social practices change the people who bring these to live. A sociogenetic account must attend to those other changes as well, or it risks completely missing the point of learning and developing as changing relations in a changing world. These and related issues are elaborated in chapter 8 and chapter 10.

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[34] Vygotsky, 'Concrete Human Psychology', 61.

5

The Intra-Intersubjective Field

In mathematics education, *sign mediation* is an important concept for describing classroom events specifically and the relation between situations and 'meanings' more generally. In philosophical terms, mediation is required whenever there are two opposing terms that cannot be connected unless there is a third thing. Classical oppositions include body and mind, an opposition that Descartes linked by mean of the pineal gland. In present-day theorizing, 'the sign' – which has a physical body and an ideal 'meaning' – takes its place. It makes the connection between the material and the ideal (non-, meta-) physical world, and therefore has the capacity to bridge between the individual subjectivities of two or more speakers. Because speakers are theorized as independent entities, caught up in their subjectivities, *inter*subjectivity becomes a problem. The sign, as defined, becomes a necessity because a gulf has been created theoretically that now needs a fix. 'The sign' is the band-aid covering up what has already been divided.

Even though much of the related research on 'sign mediation' is grounded in Vygotsky, the position is a misrepresentation of Vygotsky's approach, especially the one he was turning to at the end of his life. At that point, he was focusing on *sense* rather than 'meaning', and he emphasized the existence of a speech (semantic) field *common* to the interlocutors. It is that field where the connection to consciousness occurs, 'because [thinking] proceeds in a struggle against tendencies with semantic fields and in consciousness of these fields (cf. inner language – creates fields – extern. Language proceeds through these fields)'.[1] Thus, the sign (e.g., the word) does not stand *between* two people but is a reality for two; and it does not stand between the activity and 'meaning' but is integral to the minimal unit required to make sense during an analysis. In the phenomenological literature, too, the 'the commonness of nature is the *first* thing constituted in the form of commu-

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[1] Lev S. Vygotsky, quoted in Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive: New Findings – Notebooks, Notes, and Scientific Journals of L.S. Vygotsky (1912–1934)', *Journal of Russian and East European Psychology* 48 no. 1 (2010), 44.

nity and the *foundation for all intersubjectively common phenomena*'.[2] In this chapter, I develop this idea of the speech field based on Vygotsky's initial hunches and as taken up by two well-known Russian psychologists interested in the implications of his Spinozist turn: Boris D. El'konin and Felix T. Mikhailov.

In much of the relevant mathematics education literature, 'intersubjectivity refers to the "common ground" or "shared understandings" that participants achieve through dialogue around joint activity, such as solving and discussing mathematical problems'.[3] Thus, 'mathematical objects do not exist independently of people, but rather are produced *intersubjectively* through dialogue and convention'.[4] Mathematics then becomes 'a series of intersubjective social constructions'. From this follows that 'learning can be described in terms of students' construction of a cognitive configurations network, with progressive levels of complexity, in line with the institutional intended configurations'.[5] As a result we obtain a learner, who has been shaped within a community or institution. That learner, 'sharing an intersubjectivity' because of his/her participation in the community, 'builds a personal cognitive configuration from his/her actions on the physical and social environment. This configuration can be represented in the material world by different systems of signs subject to certain rules (syntactic, semantic and pragmatic) conveyed by the language and agreed by intersubjectivity (epistemic configuration)'.[6] This description, according to its authors, shares essential features with constructivist theories, where the individual constructs her own mind as a result of acting in a world forever on the other side of the experiential interface. Recognizing that there may exist some common foci, intersubjectivity is said to be 'begin[ning] with shared reference and mov[ing] toward shared meaning … [so that] the *growth of understanding* can be defined as the creation of greater intersubjectivity with more knowledgeable others'.[7]

The foregoing quotations show, first, that intersubjectivity is an issue that the mathematics lesson is to overcome. If intersubjectivity becomes a problem, it is precisely because the theory begins, as stated above, with the individual subjects and their subjectivities. These differ because they are located in different material bodies. From 'a constructivist point of view', the subjectivities then are divided, physically, and, therefore, separate, as they 'always remain', as everything else in the material world, '*on the other side of our experiential interface and on the other*

[2] Edmund Husserl, *Gesammelte Werke Band 1: Cartesianische Meditationen und Pariser Vorträge* (The Hague: Martinus Nijhoff, 1973), 149.

[3] Erin Elizabeth Turner, Higinio Dominguez, Susan Epson, and Luz Angelica Maldonado, 'Latino/a Bilinguals and their Teachers Developing a Shared Communicative Space', *Educational Studies in Mathematics* 84 (2013), 349.

[4] Vicenç Font, Juan D. Godino, and Jesûs Gallardo, 'The Emergence of Objects from Mathematical Practice', *Educational Studies in Mathematics* 82 (2013): 97–124.

[5] Font et al., 'Emergence of Objects', 113.

[6] Font et al., 'Emergence of Objects', 122.

[7] Susan B. Epson, 'Equal Sharing and Shared Meaning: The Development of Fraction Concepts in a First-Grade Classroom', *Cognition and Instruction* 17 (1999), 286–287.

side of our sense organs'.[8] The same take is articulated in the enactivist literature, where, 'according to Maturana, the cognizing organism is [considered] *informationally closed*'.[9] In this autopoietic project, the highest that the constructivist mind can achieve is the 'corroboration of "Others" which the subjective observer can construct within his or her own experiential domain'.[10]

The quotations also show that intersubjectivity itself becomes the purpose of the mathematics lesson – rather than, for example, the outcome of the *mathematical* activity, such as finding the solution to some puzzle (e.g. how to divide 9 pies among 12 individuals). If, on the other hand, the mathematical event becomes our organizing center – i.e. *activity* as the name of the theoretical category and the unit of empirical analysis – then it is apparent that intersubjectivity always is the condition of any joint labor that occurs. If there were an issue concerning (mis-) understanding, such as when trouble arises as a group of individuals orients toward producing what they decided or have been asked to produce, then it is resolved as a matter of course. We then observe what conversation analysts refer to as 'conversational *repair*', the purpose of which is to get the trouble resolved and the activity back on track. Without intersubjectivity, nothing would occur. We could not even speak of a mathematics classroom only of a collection of monads.

This chapter shows that *the intersubjective and interpsychological is not the problem, it is there, in public, for everyone to see*; the *intra*subjective and *intra*-psychological sphere is a problem – but only if it is disconnected and regarded separately from the intersubjective. The quandary resolves as soon as we realize that a child, from its birth, participates with others in the events of life and, therefore, is participating in its intersubjective constitution from the beginning. Any intrasubjective that exists is itself born in our participation with others – the point of sociogenesis developed in chapter 4. 'But here, the sign is by no means a mediator: it is not a mediating link in the behaviorist S–R model'[11], that is, in the way that Vygotsky himself during his earlier years had presented it. Instead, the sign is a prop of the will, 'the subjective reality of an *inner voice*, born of its *externalization* for the Other, and thus also for oneself as for the Other within oneself'.[12] Intersubjectivity makes it that we not only recognize when we are aligned with others but also when others apparently are not aligned with and why this is so. Intersubjectivity is the condition for researchers to recognize anything like a 'misconception' or 'alternative framework', notions that mark that there is something recognizable and recurrent in the talk of the other but that it does not conform to some, often implicit and unstated standard.

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[8] Ernst von Glasersfeld, 'Facts and the Self from a Constructivist Point of View', *Poetics* 18 (1989), 440.

[9] Ernst von Glasersfeld, 'Die Unterscheidung des Beobachters. Versuch einer Auslegung', in *Zur Biologie der Kognition* ed. Volker Riegas and Christian Vetter (Frankfurt: Suhrkamp, 1990), 281.

[10] von Glasersfeld, 'Facts and the Self', 443.

[11] Felix T. Mikhailov, 'The "Other Within" for the Psychologist', *Journal of Russian and East European Psychology* 39 no. 1 (2001), 26.

[12] Mikhailov, 'The "Other" Within', 17.

Historical-Philosophical Considerations

Constructivism, as other philosophical approaches, implicitly or explicitly takes the individual as the unit of analysis. This is a fundamentally Cartesian take based on the fact that in the attribute of extension, each part is outside of any other part – the philosophical *partes extra partes*. A radically different take is articulated in a Spinozist-Marxian approach, where society is taken to be the fundamental unit where the essence of human nature is to be found. As a result, there is an inherent intersubjectivity underlying anything that is characteristically human. In other words, what is human is so because it is social, intersubjective. 'The *essence* of man is encompassed only by the community, in the *unity of man with man* – a unity that is only grounded in the *reality* of the *difference* between I and thou'.[13] Ludwig Feuerbach, whose works inspired K. Marx and F. Engels in their move toward materialism, thus writes that truth only exists in and as the totality of human life and human being. But in his Sixth Thesis, Marx accuses Feuerbach of characterizing the essence of man as an abstract property inherent in the single individual, who is aware of the totality of knowledge existing at a societal level. Marx counters that the human essence lies in the totality of societal *relations*. Thus, specific about humans are not things, not shared attributes inherent in all individuals (as a Kantian approach to the general would require), but precisely the relations that bind humans together in a society of their own making.[14] 'Language', one of the most important aspects of human culture, then '*is* the practical, for other people existing, and therefore for myself existing, practical consciousness'.[15] Language, as consciousness, only arises from the need of the relation with others.

   Basing a theory on society and the societal relations as constitutive of consciousness takes on a particular light in a Spinozist take on thinking and consciousness. The Spinozist position of the thinking body taking the shape of the external form appears to suggest that all human beings should develop and have the same idea. But this is not at all the conclusion of Spinoza. Instead, the idea is based on the movement of the body. Differences arise because 'everyone's judgment is a function of the disposition of his brain, or rather, that he mistakes for reality the way his imagination is affected'.[16] Two very different things tend to get mixed up: the form of our body, on the basis of which the thinking body forms its ideas, and the things outside it. In the course of its biography (history), the movements of every thinking body come to be formed differently, leading to differences in the ideas that they form. The problem arises in part because the thinking body knows itself only through the things outside while knowing little to nothing about

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[13] Ludwig Feuerbach, *Sämmtliche Werke Band 2* (Leipzig: Otto Wigand, 1846), 344.

[14] In the Kantian approach, Adolf Hitler and Josef Stalin are not likely to have much or anything in common with Mahatma Gandhi and Mother Theresa. But all four are humans. So the human *essence* has to be based on something other than shared properties. The Spinozist and Marxian dialectics allow for the manifestation of the same thing in multiple and even contradictory terms.

[15] Karl Marx and Friedrich Engels, *Werke Band 3* (Berlin: Dietz), 30.

[16] Baruch Spinoza, 'Ethics', in *Complete Works* (Indianapolis: Hackett Publishing, 2002), 242.

the particular present movements (of the brain) that are occurring as (and shaping) thinking while it occurs.

Whereas many of Vygotsky's earlier works can be read as taking the individual to be the unit from which the social is built, and, therefore, about the role that mediation plays in overcoming the subjectivity of the individual person, he changed his position near the end of his life. Thus, whereas he wrote during his earlier career about semiotic mediation in subject–subject communication, '*in his last, "Spinozian" works the idea of semiotic mediation is supplanted by the concept of the intersubjective speech field*'.[17] Each form of human activity characterizes the types of social-material fields in which humans engage in joint labor. That field integrates over all the things that appear within it. Thus, any 'tool of labor, like language, is inconceivable outside a common field of purpose and sense, a real–ideal field of *con-sent* and *sympathy*, in the joint definition of a common action'.[18] Here, it is important to notice the adjective 'common', for what matters to joint activity is what is common rather than that which might be said to be individual – unless it has some bearing on the unfolding event, that is, unless it has, *transactional relevance*. Noting the untenable nature of the definition of language as a 'means of communication', V. V. Bibikhin notes that it 'before all else and in its initial essence, is already present in communication, and it is through communication that the act of communication acquires sense. … language in its essence, or news, is the environment in which man's historical being is realized'.[19] That is, language, as environment, is common to human beings, and language constitutes the key to consciousness – according to both Marx and the late Vygotsky.

Any aspect of the 'field [of consciousness]' is common, or it is irrelevant to the unfolding of *social* events. This is so because, as Vygotsky notes in one of the last texts that he prepared for publication, 'in consciousness, the word is what – in Feuerbach's words – is absolutely impossible for one person but possible for two'.[20] Throughout his work, Vygotsky repeatedly refers to the German materialist philosopher Feuerbach, who also was a specialist on Spinoza. Feuerbach's writings were to be a guiding thread to the development of materialist psychology.[21] His work concerns, in part, the dualism of mind and body, against which he argues in many places, including a text 'Against the Dualism of Body and Soul, Flesh and Mind'. In the immediately preceding text, Feuerbach discusses the idea of an all-knowing god. He suggests that although individual knowing and doing is limited, humans

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[17] Felix T. Mikhailov, 'Problems of the Method of Cultural-Historical Psychology', *Journal of Russian and East European Psychology* 44 no. 1 (2006), 35, emphasis added.

[18] Felix T. Mikhailov, 'The "Other Within" for the Psychologist', *Journal of Russian and East European Psychology* 39 no. 1 (2001), 26.

[19] V. V. Bibikhin, in Felix T. Mikhailov, 'Problems of the Method of Cultural-Historical Psychology', *Journal of Russian and East European Psychology* 44 no. 1 (2006), 29.

[20] Lev S. Vygotsky, 'Thinking and Speech', in *The Collected Works of L. S. Vygotsky. Vol. 1: Problems of General Psychology* (New York: Springer, 1987), 285.

[21] See Lev S. Vygotsky, 'The Historical Meaning of the Crisis in Psychology: A Methodological Investigation', in *The Collected Works of L. S. Vygotsky, Volume 3, Problems of the Theory and History of Psychology* (New York: Springer, 1997), 233–343.

do and know together. That is, the idea of divine knowledge gathers together everything knowable in a single individual; this idea is realized in the knowledge of the species. This idea has a Spinozist origin, and we also find again in Marx. After giving some examples, Feuerbach then writes the phrase that Vygotsky paraphrases for his own purposes: 'What is impossible for one person, is possible for two'.[22] Even jotting notes into a diary or reading them, Vygotsky suggests elsewhere, is a form of behavior that initially involved another person and now is directed by the person to the other in itself.

Long before Vygotsky, Feuerbach developed this idea in the context of a mathematical proof. Even if there is only one person doing a proof, 'there are two in proving; in proving, the thinker divides into two; he contradicts himself; and the thought is proven only when it has held up to and overcome this self-juxtaposition'.[23] Its purpose, so the philosopher, is nothing more than showing that something exists not only in thought. Such a proof cannot occur in thought alone. This is so because when the object of thought is to be augmented by being, then thinking itself has to be augmented by something other than being. Referring to the example Immanuel Kant used to distinguish between thinking and being – the thought of 100 thalers compared to 100 real thalers in the hand – Feuerbach points out that the former only exist in the head and thus for the individual, whereas the latter also exist for others, as the thalers can be felt and seen. He concludes that 'only exists what is at the same time for me and the Other, wherein I and the Other agree, what is not only mine [mein] – what is general [allge*mein*]'.[24] The German word for general or common literally translates as 'mine of all'. It is in and through language that consciousness is common to all; intersubjectivity always and already exists with subjectivity. Intersubjectivity is a condition for subjectivity to arise, and intersubjectivity itself is the result of the *being-with* others. Proof, therefore, is something that exists at a collective level. With Vygotsky (in this following Marx) we would say that it is a *societal* relation that eventually exists in the form of what individuals do. The adjective societal [*obšče*stvennyj], as the Spinozist-Marxian philosopher Evald Il'enkov keenly observed, is appropriate because it represents the universal [*obšče*e].[25]

Vygotsky repeatedly cites (quotes) Georg Lichtenberg, who wrote, 'one should really not say "I think" but "it thinks"'.[26] He uses it in the context of discussing two aspects of development, one pertinent to the individual the other to the collective, the personal and the impersonal.[27] There are two aspects to human behavior,

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[22] Feuerbach, *Werke Band 2*, 283.

[23] Feuerbach, *Werke Band 2*, 206.

[24] Feuerbach, *Werke Band 2*, 308.

[25] Evald V. Il'enkov, 'O vseobščem [On the Universal]', in *Filosofija et kul'tura* (Moscow: Politizdat, 1991), 320.

[26] Quoted in Feuerbach, 'Wider den Dualismus' in *Sämmtliche Werke Band 2* (Leipzig: Otto Wigand, 1846), 353. Given that Vygotsky quoted different sentences from the same page in Feuerbach, he may have taken the Lichtenberg quotation from there.

[27] Lev S. Vygotsky, *The Collected Works of L. S. Vygotsky, Volume 4, The History of the Development of Higher Mental Functions* (New York: Springer, 1997), 59.

arising from the fact that we actively participate in making material life what it is and that I change my behavior and myself through the modification of the social and material environment. Vygotsky's example is the knot in the handkerchief, which reminds me of a given errand – it 'remembered' the errand for me, it reminded me that there is something I was not to forget. To understand thinking, in this situation, we have to begin with the analysis of the societal and material conditions in which thinking arises as a response. Thus, 'a deterministic analysis of psychic life cannot begin by ascribing to thought a magical power to determine human behavior, a power to determine behavior through one of the individual's own inner systems'.[28] For people doing something together, speech, because of its materiality, is something that they have in common. It exists in their respective environments and, thus, in their respective pereživanie [experience]. It is part of the common field of activity, both conditioning subsequent speech and being constituted by the speech that occurs. The sound that rings in the mouth and vocal cords of one person simultaneously rings in the ear of the other.

The Transactional Field

In this section, intersubjectivity and the transactional field are investigated in the context of the joint labor of two pre-service teachers (Alycia and Rosy) in the process of doing a task that presented them with data. Important to the analysis are not the structures (signs) that a researcher might see in the task (Fig. 5.1) but those that the participants themselves make apparent to each other in and through their talk. This talk, therefore, serves as a means to accent the visible, that is, to allow something to show itself from itself. At the beginning of the twentieth century, researchers in very different disciplines grabbled with the fact that the same material situation as described by the physical sciences has very different characteristics when considered in terms of different organisms. Thus, the same tree trunk and its bark have very different objective characteristics, affording very different types of actions, in the life of a pine beetle, woodpecker, or human being. The tree trunk thus constitutes a very different form of environment or 'Umwelt'. In the human sciences, the term 'Lebenswelt [lifeworld]' and *field*, as employed in phenomenological sociology, Heideggerian artificial intelligence, and Gestalt psychology, respectively, have gained currency. A *field* is the ensemble of things perceived and acted upon, and therefore relevant to, what is done in a particular social situation.[29] There is therefore a whole–part relation between the things salient in and to the participants and the kind of societal material situation in which they find themselves. Thus, for example, we may not speak of the nature of the inscriptions in the following analyses independent of the overall activity, which may be glossed as

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[28] Vygotsky, 'Thinking and Speech', 50
[29] See, for example, Aron Gurwitsch, 'The Field of Consciousness', in *The Collected Works of Aron Gurwitsch (1901–1973) Volume III* (Dordrecht: Springer, 2010), 1–409.
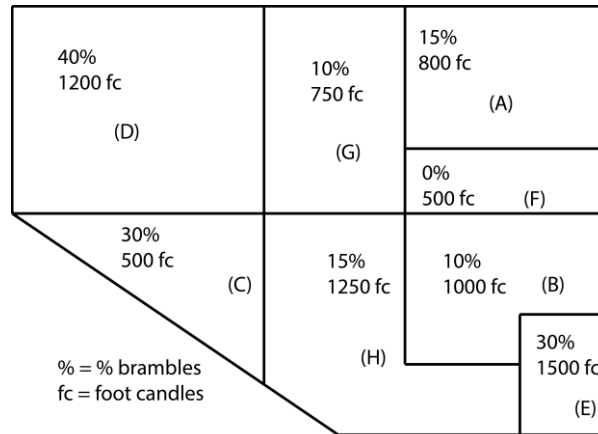
Fig. 5.1   The task presents a map with the data that an eighth-grade student collected on the bramble density and light intensity. After the situation, the stem asks '(a) What patterns, if any, do you see? (b) What claims would you make? and (c) How would you support your claims?

'participation in a social science research project focusing on the analysis of data'. What is salient to and made salient by Alycia and Rosy is so in view of their voluntary participation in the research project. What they do, therefore, will not (necessarily) be the same that they might do in a science course, mathematics course, or as members of an environmentalist organization. Even though this might sound counterintuitive, research actually shows that a middle-school student may act consistent with his 'learning disability' label when asked to interpret data in the mathematics class; but he teaches peers from another class how to analyze the same kind of data in the context of a science class.[30] Thus, a field encompasses all those things perceived and constituted as pertinent to the present activity, where 'activity' is taken in the cultural-historical sense of the process of the generalized production for collective need satisfaction.

A Case of 'Doing [Analyzing Data]'

In this section a fragment is provided in which two pre-service teachers participating in a research project on data analysis and graph interpretation are in the process of attending to a task that asks them about patterns seen, makeable claims, and evidence (Fig. 5.1). The fragment is subsequently used to show that the concern for intersubjectivity may be inappropriate given that the participants are oriented toward completion of a task rather than toward the 'construction' of 'meaning'. The

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[30] Wolff-Michael Roth and Angela Calabrese Barton, 'Constructing Scientific Dis/Ability', in *Rethinking Scientific Literacy* (New York: Routledge, 2004), 129–155.

participants act within a field, of which they and their mutual other are part, and in which they make use of all sorts of things and features *for the purpose of* doing what the task specifies. The work involves making available structural properties, through hand/arm movements that can be seen as following or pointing to features of a field that is and is shown to be common. Language, as symbolic movements, is not so much *about* something but essentially has the function to constitute the accented visible. Language, as symbolic movement, has what philosophers have called an apophantic function, that is, the function of allowing some phenomenon to show itself from itself[31], *for the purpose of getting the job done*. In addition to the function of getting the job done, language, as a material thing, also *is* the person–person relation.

The fragment derives from a study in which future science teachers were asked to complete a task that eighth-grade students previously had completed in another study. The task featured a map on which bramble densities and light intensities for different plots had been recorded. The accompanying, real-life-based story attributes the data to an eighth-grade student who had lost her data, leaving it to the participants to figure out whether there are any patterns in the data, the claims that could be made, and the evidence that could be identified in support of the claims (Fig. 5.1). Although all the participants in the first part of the study were in their fifth, professional year, which followed the completion of a bachelor's degree in science or mathematics, the results showed that pairs of eighth-grade students produced more abstract inscriptions with a higher frequency than the future teachers – at a level that was statistically significant.[32] The second part of the study – designed to investigate just where any trouble for pre-service teachers might arise – had pairs from an elementary education program with science and mathematics emphasis do the task while requiring pairs to produce one solution that they both could agree upon. Alycia and Rosy, who figure in the following fragment, were two of the participants. The following fragment of the *protocol* of their joint labor concerning possible patterns between plots C, D, G and B.

Fragment 5.1


```
01   A:   twelve hundred and then we hav:e thir-
          ty ((points to C)) hh here ((C and E))
02        (1.2)
03        and two of this aside to it, so it doesnt
          show location either ((writes numbers
          below map))
04        (4.9)
05   R:   >whatd'you-mean< it doesnt show loca-
          tion.
06 A:     well
```

[31] Martin Heidegger, *Sein und Zeit* (Tübingen: Max Niemeyer, 1977), 32.

[32] Wolff-Michael Roth, Michelle K. McGinn, and G. Michael Bowen, 'How Prepared are Preservice Teachers to Teach Scientific Inquiry? Levels of Performance in Scientific Representation Practices', *Journal of Science Teacher Education* 9 (1998): 25–48.

07   (3.3)
08   shes doing research on the ecozone ((*points to text*)) but (0.6) I guess shes
        just finding out doing the density and brambles.
09   (0.5)
10 R: yea
11   (0.2)
12 A: an different areas ((*points to text*)) of her (0.2) ecozone (.) it doesnt say
        much ((*rH movement over map*)) (.) like (0.6) we dont say like this is in
        the left ((*lH movement over bottom of map, emphasizing left then right*))
        hand corner or right
13   (0.4)
14 R: yea
15   (0.2)
16 A: corner (0.2) we don'know h*ow* shes
17   (0.4) ((*Rosy points to G, then D*))
18 R: uh hmm
19   (1.6)
20 A: like if this is [*al*l *one e*cozone]
                        [((*circles map*))]
21   (0.8)
22 R: .he >it just [doesnt even occur] I mean< if this ((*D*)) was
                     [((*B to D to B*))   ]
23   (0.8)
24   all ((*A to G to D*)) even with the whole area thing doesnt make sense ei-
        ther because (0.3) this ((*G*)) should be d uh ((*G*)) (0.3) with this ((*D*)) one,
        if theres a pattern between these ((*D*)) two ((*G*)) ((*G to D*)) .hhh there
        should be three hundred ((*circles plot G*)) right? (0.2) >because thats
        ((*D*))< (.) >forty=percent< (0.2) in tw*el*ve hundred
25   (0.9)
26   well ((*G*)) it won'doesnt make sense there.
27   (0.9)
28   and then [G]
29   (5.4)
30 A: °°what'd u°°
31   (1.3)
32   [do you mean] [ac*ross*   ] (0.2) [of?]
        [((*D*))   ] [((*D to B*))]   [   ]
33 R: [yea].

    The episode might be glossed in this way. Alycia and Rosy are looking at pairs
of plot at a time to see whether there is a pattern. At one time, Alycia also com-
ments that 'location' is not shown, which becomes problematic with the next turn
apparently asking, following a considerable pause, what she means (turns 03, 04).
A similar situation ends the fragment, where – after Rosy points to the plots D and
G, articulates their data pairs, and says that 'the whole area thing does not make

sense' – we find a turn treated (because of the rising intonation toward the end) as query, 'do you mean across of?' We even find the phrase 'I mean' within a phrase, where it relates 'it just doesn't even occur [to me?]' and 'all, even with the whole area thing, doesn't make sense either'.

Analysis

In the fragment, there are three instances where we find a phrase including the verb 'to mean'. But taking any one of these instances as a rationale for turning research into a question of intersubjectivity rather than an analysis of the activity, the current project, as a whole is going to miss comprehending what the participants are doing. Consider the following excerpt from the fragment including the turns preceding and succeeding the 'mean'-containing phrase. What kind of joint work are we observing? First, there is a {statement | query} pair (turn 03 | turn 05}.

Fragment 5.1.1
03   and two of this aside to it, so it doesnt show location either ((*writes numbers below map*))
04   (4.9)
05   R: >whatd'you-mean< it doesnt show location.
06   A: well
07   (3.3)
08   shes doing research on the ecozone ((*points to text*)) but (0.6) I guess shes just finding out doing the density and brambles.

According to the method exhibited in Fig. 3.10, we see that 'it doesn't show location' cannot be attributed to a single individual, Alycia.

Fragment 5.1.2
A: ((*says*)) it doesn't show location either
R: ((*hears*)) it doesn't show location either   ((*says*)) what do you mean, 'it doesn't show location'
                                                  ((*hears*)) what do you mean, 'it doesn't show location'

Instead, 'it doesn't show location' also belongs to Rosy, not in the least because she articulates the phrase in turn It is possible to identify many studies in the literature where such exchanges are taken as data to claim that there is an initiation of the construction of 'intersubjectivity'. But what the two are doing here would be impossible if intersubjectivity did not already exist. They can do what they do precisely because there is intersubjectivity. Rosy hears a phrase, and she repeats it. In taking up the phrase, making it the topic of a phrase on her own, she treats the

foregoing as a move toward a project, but as one that does not in itself make apparent its pertinence. Rather than taking what happens as something in itself, we are better off focusing on what the two are after, which is the production of answers called for in the task description. That pursuit (project) requires labor. We can then see turn 05 as a move that makes this labor itself the topic, which is apparent when the phrase is glossed as, 'How does what you say advance us toward the end product?' Rather than 'meaning', the turn can be heard as asking for a restatement that is such that it evidences (from itself) its pertinence to the ongoing work. That is, what that unfolding give and take produces is not some transcendental, metaphysical (i.e. immaterial) 'meaning' but another, more extended phrase. It is as if Rosy had asked, 'what do you mean to say [by] "It doesn't show location"', which is paired with the reply '[I meant to say], "She is doing research on the ecozone, but I guess she's *just* finding out doing the density and brambles"'. That is, the reply part is no less material than the invitation part, nothing otherworldly, nothing abstract, nothing that could not itself be pointed to again – where 'pointing' is achieved in this situation prosodically and, in this text, by means of the double quotation marks.

As the unfolding of the excerpt shows, turn 08 treats the preceding as a query, producing a reply in which the issues in the task are stated as being limited to the density and light intensity, thereby also phrasing that other issues, elaborated more in subsequent turns, are not at stake. This includes precisely where the data was collected in the zone as a whole or in each of the eight plots. Again, there is intersubjectivity, as there are turn pairs of recognizable form. The whole problematization of intersubjectivity becomes problematic as soon as we take social phenomena as phenomena in their own right. If we had approached Rosy and Alycia without previously knowing what they were doing, we could have seen it; and we might have said something like, 'They are trying to figure out a pattern in some data'. But we can gloss an event in this way only because we see, in the concerted effort, this rather than another form of event. The gloss points us to the social nature of the phenomenon. Social phenomena, by the very definition of the term, can be considered like things that are irreducible to single individuals and their psychology.[33] But that irreducibility is seldom heeded, which led the well-known Vygotskian psychologist Vasilii Davydov to complain: 'though acknowledging the role of Vygotsky's ideas on collective activity, [some scholars] nonetheless go on to reduce collectivity to the simple sum of the actions of a group of individuals'; and, in so doing, 'each person is seen as being a priori autonomous'.[34] Here, I shall thus take a unit of analysis that encompasses the collective work of Alycia and Rosy – which is the beginning of Vygotsky's model of the formation of consciousness captured in the formula 'collective activity – culture – the ideal – sign or symbol – individual consciousness'.[35] This reflects the fact that anyone walking into the

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[33] Émile Durkheim, *Les règles de la méthode sociologique septième édition 7ième ed*. [Rules of Sociological Method 7th ed.]. Paris: Felix Alcan, 1919), 20.

[34] Vasilii V. Davydov, 'L. S. Vygotsky and Reform of Today's School', *Journal of Russian and East European Psychology* 36 no. 4 (1998), 93.

[35] Davydov, 'Reform', 92–93.

Fig. 5.2 Because their joint labor is a recognizable social fact, Alycia, Rosy, and their material environment (including the task) constitute the unit of analysis. Alycia is part of Rosy's environment, and Rosy is part of Alycia's environment
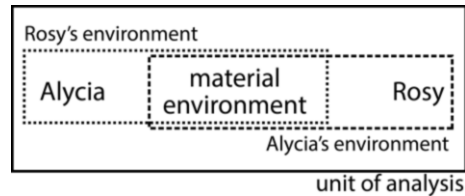
room, seeing the camera, its operator, the two women, the sheets in front of them might have said, 'They are collaborating on a task for a research project'. We may depict this unit diagrammatically (Fig. 5.3). For Alycia, Rosy, the task, and all words articulated are part of the environment; for Rosy, Alycia, the task, and all words articulated are part of the environment. Because their joint labor is a recognizable *social* fact, Rosy, Alycia, the task, and all words articulated are integral parts of the unit of analysis. As parts of a whole they cannot be understood independently of this whole and, thus, independently of any other part or ensemble of parts. Both individuals are part of the field. Not only do they constitute and make salient to each other aspects of the field but also they make salient the very ways in which this constitution operates. This gives us an inroad to intersubjectivity that apparently is a problem for the constructivist approach.

Intersubjectivity is an irreducible 'property' of the joint labor, which, here, is the analysis of data for the purpose of producing data for a research project on data analysis. The issue therefore is not *what* is said, and the production of 'intersubjectivity' or 'intersubjective meaning'. Thus, '*"shared agreement" refers to various social methods for accomplishing the member's recognition that something was said-according-to-a-rule and not the demonstrable matching of substantive matters. The appropriate image of a common understanding is therefore an operation rather than a common intersection of overlapping sets*'.[36] Here, the matching would be between some sign – or rather signifier, if 'sign' is understood as the signifier–signified relation – with a referent (signified). Once the idea is dropped that speaking is primarily a matter of matching sign and referent, signifier and signified, then we can take 'the recognized sense of what a person said [to] consist only and entirely in recognizing the method of his speaking, of *seeing how he spoke*'.[37]

That the recognition of what is being done has precedence over any matching of words and meaning is seen from the fact that the very statement 'what do you mean' is taken to have a proper place in the conversation. That is, this recognition is invisible because it is treated as a normal thing to do under these circumstances. Thus, when articulated in other circumstances, its very production comes to be an issue. This was the case when Garfinkel gave his undergraduate students the assignment to ask an interlocutor to clarify the sense of some common, everyday statement. In one instance, a person said, 'I had a flat tire', to which the undergraduate student replied by saying, 'What do you mean, you had a flat tire?'. The origi-

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[36] Harold Garfinkel, *Studies in Ethnomethodology* (Englewood Cliffs: Prentice Hall, 1967), 30.

[37] Garfinkel, *Studies*, 29.

nal speaker seemed stunned, apparently replying in turn by saying, '"What do you mean, «What do you mean?» A flat tire is a flat tire. That is what I meant. Nothing special. What a crazy question"'.[38] Readers immediately see the parallel to the present situation, as the opening turn pair has precisely the same structure as (turn 03 | turn 05). However, whereas the turn 'what do you mean?' is an appropriate part of the work that Alycia and Rosy do, it is treated as inappropriate in Garfinkel's case. That mutual recognition of the appearance of 'what do you mean?' is itself integral to any intersubjectivity, without which what the two women are doing becomes incomprehensible. That is, if we take Alycia's perspective for an instant, then it is apparent that she treats the question as legitimate, that it is something that can be expected and that it is to be treated in an appropriate manner. That is, social actors conduct their everyday affairs, here the participation in a research project, such that they talk, the ones for the others as for themselves, in plain, ordinary, reasonable and reasoned language. These ways for the most part are recognized and unproblematic.

Ways of talking, gesturing, and doing things are such that they maintain the conditions for talking, gesturing, and doing things in those same ways. When something appears irrational or unreasoned, then it is better to regard it as an issue of unknown ways rather than reasoning in and of itself. Thus, 'persons require these properties of discourse as conditions under which they are themselves entitled and entitle others to claim that they know what they are talking about, and that what they are saying is understandable and out to be understood'.[39] Words, language, are part of the field, and so are the speaking pauses, resources for doing things. But all too often concerns for words and what they are taken to stand for become too important, often getting out of proportion to such an extent that researchers lose sight of other phenomena that are equally or more important and that could teach us a bit about the real life of language. Let us take a look at hand and arm movements.

As thinking bodies, we are part of each other's fields – things, movements, positions, sounds, smells, and so on. Anything in the field may have some explicit function in the ongoing event. Other potential things in the field are not thematic – though nevertheless part of the ground that exists with the field. Thus, for example, the paper on which the task is noted, such as its whiteness or size, apparently is not at issue in the fragment or in fact at any time during the entire session. It nevertheless is part of the ground that allows something like the map and the typed word to appear. Importantly, some pencil or finger positions or movements are treated as relevant and even necessary for the event to unfold, whereas others are not treated in this manner. From the analyst's point of view, the relevance of a movement or position can be recognized from the way in which participants themselves treat them; and another, more indirect way is available in the ways that unsuspecting transcribers render the contents of a video. Thus, the transcriber of the present tape would note, in the case of Fig. 5.3, that pointing to plots C and F occurred; simul-

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[38] Garfinkel, *Studies*, 42.
[39] Garfinkel, *Studies*, 42.
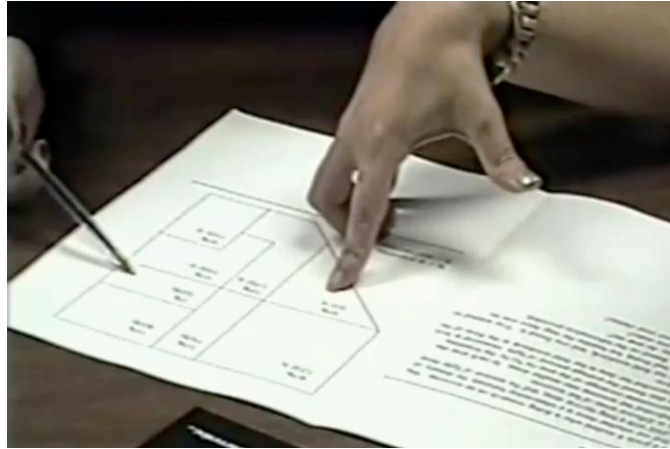
Fig. 5.3 The map, numbers, and words on the page constitute part of the *intersubjective* field; other parts include the mutually available bodies, and the sound-words

Fig. 5.4 Some hand or pencil movement and positions are taken semiotically – including the finger position in turn 01 of Fragment 5.1 and the mutual pointing gestures in Fig. 5.2 – whereas others, such as the pencil movement here, are not taken in this way. The distinction between relevant and non-relevant movements is itself integral to competent practices of practical intercourse with others
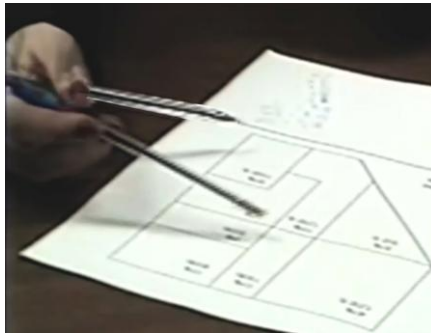
taneously, the transcriber did not indicate a relevant movement in the case of Fig. 5.4. That is, one particular finger position was noted to constitute pointing or gesturing, whereas the other was not. We also observe that Alycia and Rosy take up the positions from each other in the apparent, and highly unproblematic case of pointing to specific plots.

What then underpins the distinction of some movement or position as an instance of gesturing from another movement or position as constituting something irrelevant? The response lies in the fact that pointing and gesturing movements, in this case, are not the proper unit. There is in fact a dialectical relation between anything that can be treated as being pointed to or gestured and the action of pointing and gesturing. Take the instant depicted in the offprint of turn 01, where the little ('pinky') finger and the index finger are placed on the plots C and E, respectively. Part of the field are the words 'we have thirty here'. Now, from the perspective of the recipient, some movement or finger-hand-arm configuration makes sense to be treated as pointing only if there is something that can be found to be the object

pointed to. But pointing, in this case, is motivated by the thing pointed to; it is a reply – in the sense of the term developed in chapter 3, the ensemble of solicitation and reply – to the presence of the thing. In a way, the thing constitutes the account of the pointing. The practical social action then may be articulated as 'doing [pointing to | finding *the thing*]'. If that joint action does not succeed, then it has the right to become an issue. Otherwise it is treated as self-evident and asking, 'What do you mean?' or 'What are you pointing to?' would be highly inappropriate that itself has the right to become an issue. Tentatively, a participant might say, 'Are you trying to point me to something?', in which case the ambiguity of a hand-arm-finger position or movement is raised. Pointing, here, is social through and through, *inter*subjective (collective) rather than *intra*subjective (individual) work. As seen in chapter 4, the emergence of (intentional) pointing as a social practice is indeed part of Vygotsky's argument concerning the primacy of the social and the sociogenesis of all higher psychological functions.

Pointing and gesturing are integral to the communicative features of the unfolding event. They play together with the semantic aspects of sound-words. Pointing and gesturing *are necessary* because without them, the communication would be underdetermined. This is apparent especially when body movements and positions come together with what are known as indexical terms, including the instances of 'this', 'there', 'that', 'here', and 'it' that are found in Fragment 5.1. Together with the finger or pencil positions or movements, their use constitutes an accented visible. In this case, the sound-word 'thirty', the written '30%', and their apparent location and association with plots C and E come to stand out. It would be quite arbitrary and useless to ask questions about the meaning of any of these accented parts of the two participants' reality or about their intersubjectivity. Their *joint* labor would deserve the adjective if all these aspects were not intersubjective – which they inherently are because of their material nature.

We do not know what might have been in their individual heads and what we might have found out if we had separated them right after Fragment 5.1 and asked questions about what they mean by 'pattern' (turn 24). We do see that the statement about the plots D and G treats 'the pattern' as 'proportion', for it contrasts the pair {40%, 1,200fc} with the pair {10%, 750fc}, specifying, while pointing to the 750fc, that it 'should be' 300[fc]. As the transcription shows, the statement is treated as going without saying, that is, without requiring further talk. It can go without further saying, among others, if the proportionality 40:1200::10:300 is taken as the ground. There might be other grounds, and, therefore, different statements what was 'meant to be said' or 'heard as said'. But any such difference does not play into the way in which the episode currently unfolds and, therefore, has no bearing on its historical trajectory.

In this entire event, we grasp nothing of the *internal* dynamic, of the give and take, unless we accept the existence of intersubjectivity. The event would not unfold in the concerted way that we see unless each word, as the whole world (field), is considered as reality for two people. This is precisely what Vygotsky was working toward near the end of his life, as seen from the statement at the end of his posthumously published *Thinking and Speech*. As noted earlier, he was working

toward the idea of the common speech field, where language constitutes consciousness for other people and therefore also for the speaker him-/herself.

Mediation

In mathematics education, the term 'mediation' has considerable currency. Its use tends to be attributed to Vygotsky, and especially to the text *Mind in Society* where there are a number of figures showing how some third entity exists between two original entities.[40] But that book is highly contested today because, as historical text analyses show, it was neither written by Vygotsky nor does it represent his ways of thinking[41], especially not the ways of thinking characteristic in the final, Spinozist period of his life. The editors of *Mind in Society* 'constructed' some of the chapters based on other texts, 'summarized' something else Vygotsky had written, 'inserted material from additional sources' 'to explicate the meaning of the text', and enacted many other changes that produced a text no longer consistent with the psychologist's work.[42] Mediation is one of those terms that Vygotsky, according to Russian specialists and those with insights into his largely unpublished personal notes, abandoned in favor of the idea of a semiotic field – an idea more consistent with the category of *pereživanie* [experience] articulated more extensively only a month preceding his death.

In the Mathematics Education Literature

The idea of mediation appears, as noted in the introduction, whenever there are two autonomous entities that differ in some aspect, for example, the teacher and a student who have different situation definitions. In this case, 'when the situation definition of a task is shared, then intersubjectivity (between the teacher and learners in relation to the task is easily established. It is when the situation is not shared (either within a group of learners or between a learner and the teacher) that mediation is required for intersubjectivity to be established'.[43] Mediation is semiotic when language and other forms of signs are implicated, such as in communicative situations or when, in a think-aloud or clinical interview session, a student uses some folk or formal mathematical notion as part of solving the current task or an instructor uses

[40] Lev S. Vygotsky, *Mind in Society: The Development of Higher Psychological Processes* (Cambridge, MA: Harvard University Press, 1978), 40, 47.

[41] See Anton Yasnitsky and René van der Veer, eds., *Revisionist Revolution in Vygotsky Studies* (London: Routledge, 2016).

[42] See Vera Steiner, Ellen Souberman, Michael Cole, and Sylvia Scribner, 'Editors' Preface', in Lev S. Vygotsky, *Mind in Society*, ix–x.

[43] Jill Adler, 'A Participatory-Inquiry Approach and the Mediation of Mathematical Knowledge in a Multilingual Classroom', *Educational Studies in Mathematics* 33 (1997), 239.

a text to plan lectures and teach. The notion of mediation is so pervasive that no mathematics teaching or learning can do without it. Thus, 'school mathematics *requires* mediation, and specifically mediation between everyday and scientific concepts – between previously acquired mathematics and new mathematics'[44] and an equivalent separation of theory and practice, which may be accomplished by technical tools. The term mediation also is applied in situations when there are differences between the ways in which a teacher works and talks *within* the classroom, and those ways that she uses when, for example, talking to a researcher *about* her classroom (work). As in the children's case, where the differences between everyday scientific concepts and scientific concepts have to be mediated to join up, the two ways of talking – i.e. within and about – require mediation, because they do not easily join up. The notion is used whenever a difference is postulated between two self-contained entities that are external to each other and, as Spinoza articulates so well, cannot affect one another. Mediation, thus, always is *mediation across differences*. The term is also sometimes used to refer to an encompassing unit, such as when it is claimed that 'activity mediates between the cultural and the social on the one hand and the individual on the other'.[45] In the Spinozist-Marxian take, activity does not mediate but instead constitutes a unitary phenomenon that manifests itself in a plurality, including that of the cultural (social) and the individual.

   One of the important differences that mediation is said to overcome is that between the inside of a person ('Self'), the *intra*psychological dimension, and the outside of the person, the *inter*psychological dimension ('Other'). But such a separation is precisely overcome in the late works of Vygotsky. The distinction between inside and outside, Self and Other, is itself the result of a continuous movement, which also is responsible for the fact that there is nothing inside that is not always already outside, as per the Spinozist position that Vygotsky worked out in his final year (see chapter 1). That movement is apparent as soon as we think of speaking in both sensual material and supersensual ideal terms. Each word spoken contributes to the material environment, changing it, and thinking, because it takes account of the environment, immediately adapts to the changes. As a result, further words are the result of having taken into account a changing environment.

   In the attempt to overcome differences, teachers are said to command a range of resources to mediate the differences and associated 'tensions' or 'contradictions' that arise in classroom talk, include 'everyday language', 'metaphors', 'gestures', 'code-switching', 'signs', '(cultural) artifacts', 'persons', 'tools', 'technology', 'textbooks', etc. Some mathematics educators even suggest that 'the activity' mediates the relations of students to the world and with others, peers and teachers alike. 'By acting as mediators, technical tools or simply tools and symbolic signs, structure human practical activity and bring into play differentiated mental pro-

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[44] Adler, 'Participatory-Inquiry', 240, emphasis added.

[45] Eric Axel, 'One Developmental Line in European Activity Theories', in *Mind, Culture, and Activity: Seminal Papers from the Laboratory of Comparative Human Cognition* edited by Michael Cole, Ÿrjö Engeström, and Olga Vasquez (Cambridge: Cambridge University Press, 1997), 136.

cesses which in turn regulate and qualitatively transform that practical activity'.[46] The difference with the position taken in the late Vygotskian texts, which develop a Spinozist-Marxian take, is apparent. This is so because in a one-substance approach, any part of an event never stands on its own: it is part of a hole. Once activity is considered *as a whole*, there are no parts external to each other. This is so because each part contains the whole, as much as the whole contains all the parts – as Vygotsky notes about the relationship between consciousness and the word, a situation in which 'consciousness is reflected in the word like the sun is reflected in a droplet of water', and where the 'word is a microcosm of consciousness, related to consciousness like a living cell is related to an organism'.[47] Thus, once a tool is used, the activity no longer is the same, it is a different activity in which the tool is an integral part, a part that is related to all other parts and the activity as a whole and, therefore, inherently *cannot be* between *any two parts because it is itself part of these parts*.

The theoretical use of 'mediation' and 'mediators' may be depicted in diagrammatic form by employing the logic and illustrations apparent in the literature and in the publications of some of Vygotsky's earlier works. Thus, in the relation between two subjects, a word (language) – a special case of a sign – and more generally anything that has semiotic value in a conversation (gesture, some mathematics related thing like a cube, graph, diagram) comes to stand between them thereby overcoming the impression that there could be a direct, *im*mediate relation between the two persons (Fig. 5.5a). In the case of a third person, such as when a teacher becomes part of the talk in a student dyad or small group, the situation may be depicted to account for her role as well. For example, some mathematics education research suggests that the teacher mediates between students who have made different statements and either cannot come to an agreement or do not understand each other. The teacher might use plain language to assist in the articulation of the matters such that the two individuals now understand the position of the other and, in this way, are in a position to exit their impasse (Fig. 5.5b). Thus, 'teacher-mediated interaction' involving two students commonly denotes a situation in which two students get to respond to each other because the teacher steps in and, through some form of invitation, gets at least one student to talk up the reply of another and addresses it in her own reply. In such a situation, the teacher is something like an intermediary, as this type of role is found in many social situations, where there are two parties that get to talk to each other only by means of, through the mediation by, a third party, the mediator. For example, one student might have a question and another one replies but the teacher has the sense that they do not understand one another. Her talk, which somehow clarifies some of the content, then is said to have mediating function. Public discussions involving the teacher and the whole class may then be referred to as 'class mediations'. The use may then proliferate, because such discussions mediate diversity in the case of multilin-

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[46] Dmitris Chassapis, 'The Mediation of Tools in the Development of Formal Mathematical Concepts: The Compass and the Circle as an Example', *Educational Studies in Mathematics* 37 (1999), 276.
[47] Vygotsky, 'Thinking and Speech', 285.
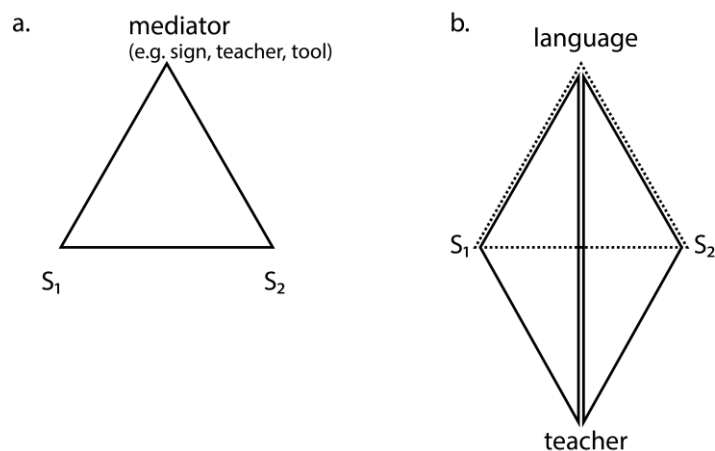
Fig. 5.5   **a** Some mediator is theorized to stand between, and bridge, the gap between two subjects S_1 and S_2 (e.g. teacher and student, student an student). **b** Depiction of semiotic mediation by the teacher, who is using language, a semiotic tool, already in use for her relation to the subjects individually, to create a bridge (dotted triangle) between the two students (S_1, S_2) who make differing statements

gual or multicultural classes. In this instance, the concept 'mediator' is used in its everyday use to describe the function of a person with respect to two individuals or parties. It is not apparent in which way this function is the same as when 'word', 'sign', or 'gesture' is modified by the adjective 'mediational'.

The use of the term 'mediation' and its verbal form 'to mediate' may slide. It then is simply used synonymously with other terms or verbs, such as 'elaborate' or 'explicate'. This is apparent in a statement such as, 'If [the teacher] does not mediate publicly how and why [one student's] response is a general answer, but [another student's] a specific case, then her intention to develop "good explanations" through pupil-pupil interactions will be thwarted'.[48] Here, the verb 'to explain' would have done the job just as well. In another situation, the teacher is said to 'mediate' students' verbal-semiotic mediation, when in fact she is 'doing [rephrasing]'. Important here also is the notion of 'interaction', which always is used when two autonomous entities somehow act upon one another. Instead of writing that the teacher mediates whole-class talk about some mathematical issue or that the teacher mediates student-student *inter*action, we might just as well say that there is teacher facilitation.

There already have been doubts about the idea that differences between student statements could be 'negotiated', where it is left open for the moment whether the mediating role is attributed to the teacher, language, task, material, or artifact. Like the term *inter*action, the notion 'negotiation' 'carries with it the vestiges of indi-

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[48] Adler, 'Participatory-Inquiry', 254.

vidualistic thinking'.[49] Deriving from the Latin *negōtiārī*, 'to do business', 'to act as a banker', and 'to trade', the term refers to situations in which two or more parties are involved in some form of reciprocal exchange. But one can only exchange something that one has – which does not explain how students, who make different statements, arrive at some third statement that overcomes the differences. This constructivist position, which was already apparent in Jean Piaget's writings, is problematic within a cultural-historical take generally and within the Spinozist-Marxian take of the late Vygotsky specifically. Thus, with the idea of mediation and the negotiations that it might involve, we end up with the portrayal of interaction 'as obeying an individualistic logic of best self-interest. I interact with other individuals, giving and receiving services, expecting to end up with increased meanings and a more robust personal way of thinking'.[50]

A Case Study

The purpose of this case study is to exemplify that 'negotiation' and the associated problematic nature of intersubjectivity is the immediate consequence of an individualistic take on social phenomena, such as the completion of tasks in mathematics lesson. If the investigator were to take a sociological orientation according to which a *social* phenomenon is so sui generis, in its own right, then it cannot be reduced to the (sum of) individual actions that somehow come to correspond in *inter*action because the nature of each action is a function of another action. In this take, a query is taken to be a query because there is a reply, and a reply is such because there is a question. Thus, queries and replies only come as irreducible pairs: {query | reply}. If a statement is not followed by a reply, it was not a question even if the speaker had intended it as such. Language philosophers, including Mikhail Bakhtin and Valentin Vološinov, have argued for quite some time that an individual phrase is nothing, and if it is a query, reply, invitation, rejection, and so on than it is so as a sociological rather than linguistic fact.[51] The *function* of a phrase in a particular event cannot be derived from its linguistic (semantic) properties.

Description. In this lesson fragment, we return to the three girls from chapter 2 tasked with finding out the nature of the mystery object hidden in a shoebox and representing it by means of shaping a blob of plasticine in its image. Melissa has shaped 'a cube', and she has explicated already twice why what she feels in the shoebox has to be a cube by giving an operational definition that states what one

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[49] Luis Radford, and Wolff-Michael Roth, 'Intercorporeality and Ethical Commitment: An Activity Perspective on Classroom Interaction', *Educational Studies in Mathematics* 77 (2011), 229.

[50] Radford and Roth, 'Intercorporeality', 229.

[51] See, for example, Mikhail M. Bakhtin, 'The Problem of Speech Genres', in *Speech Genres and Other Late Essays* (Austin: University of Texas Press, 1986), 72.

has to do to know an object is a cube. Jane and Sylvie, on the other hand, are shaped differently, though all three have shaped their plasticine into rectangular solids. About 10 minutes into the task, one of the two teachers in the room, Mrs. T., comes to the table and speaks to them. Just before the following fragment begins, she has solicited from the three children what the mystery object is, as they have expressed it in their plasticine models, and as received or provided names for them ('cube' and 'rectangular prism').

Fragment 5.2a

147 T: bt lets take a [look] ((*lines up all 3 objects*))
   a   (2.2)
   b   y'know theres like different shapes of things (0.3)

   c   >well [th*is* is a cu]be (.) ((*Melissa pulls cube toward herself*))

   d   and [th*ese* are] rectangular prisms. right?
148 M: >°°hm heh°°<
   a S: °yea°

149 T: but its [*on*ly *on*e object in here
            [((*reaches into the box, S, J, and M gaze at her*))

   We note that Mrs. T. first gathers all the objects, lining them up, so to speak, making them visible in one and the same glance (turn 147). The opening word is the oppositive conjunction 'but' that announces and prepares what is to be heard as being in opposition to what has been said or happened before. The three models have different shapes (turn 147b), or rather, as the self-correction states: there is a cube (147c) opposed to rectangular prisms (turn 147d) gathered together both by the plural pronoun 'these' and by their perceptual grouping, which is enhanced by bringing Jane's and Sylvia's models even closer. We hear a brief laughter and an acknowledgment before another oppositive conjunction introduces the fact that there is only one object in the box. In plain view of the three girls, Mrs. T. reaches into the box, searches for the object, and states that she found *it* (rather than *them*), thereby confirming that there is only one rather than several objects, which, by implication could have different shapes.
   There is an invitation to come to an agreement on whether the mystery object 'looks more like *this*' Sylvia's, Jane's, or Melissa's model, while Mrs. T. grabs and holds up each one of the three objects. She states, captured in Fragment 5.2b, what can be heard as an instruction for producing agreement: come together and decide

what it is; and she then adds what can be heard as a rationale grounded in the earlier observation that there is only one object in the box. The instruction comes to be repeated (turn 156), preceded by an affirmative adverb ('yea') and is followed by an opening offer for the agreement to settle on the shape that Melissa has produced (turn 157).

Fragment 5.2b
154 T: >you have to come together and dec*ide* what its gonna be because its
  [only *on*e *ob*]ject in here.< right? ((*gazes at Sylvia*))
  [((*hits box 3x*))]
 a (0.5)
155 S: yea=
156 T: =and >as a group ye have te say< *we* th*ink* it=*i:is* (.) becau:se;
  [okay?
  [((*begins to get up, and then leaves*))
157 M: i'm r*eal*ly thinking its a cube ((*gazes at her model*))

In the six minutes that follow, the girls eventually establish a procedure that is performed twice (Jane, Melissa): one hand outside the shoebox and visible to all feels one of the models, turns it, feels the surface, turns it, etc. The other hand is inside the shoe apparently doing the same simultaneously. As the result of what happens, the group settles on the mystery object to be a rectangular prism similar to the model Jane has shaped.

Analysis. The situation described is a typical candidate for invoking the notion of mediation and intersubjectivity. Following a tactile exploration of an object in a shoebox, the three girl produce different models thereof. That is, whatever is in the shoebox has felt differently, a sense that is apparently reproduced when each individual holds the model she has shaped. Some investigators might employ the term 'meaning [of the object]' that each participant has constructed. There are then three intrasubjective 'constructions' that somehow stand in opposition. In this take, the teacher now has to help them, functioning as a *mediator* to overcome their subjectivities and to produce intersubjectivity by coming to an agreement. Because they initially disagree, the process that follows and leads to the agreement might then be called 'negotiation'.

Whereas such a description may sound reasonable, there is a big problem with this description, which turns out to be substantial and deadly: it is based on an individualistic ontology. This ontology leads to logical contradictions in the theory of the same type as the one that pre-Marxist philosophers of political economy encountered trying to theorize economic exchanges. Thus, the traditional take was that the use-value of a commodity and its exchange-value are the different perspectives of seller and buyer. When these individuals 'negotiate' the relative amounts of the two commodities exchanged in a barter, a third phenomenon enters – which historically will eventually become 'money' – that serves as a relative measure of the value of each of the two substances in a barter exchange while, in early econo-

mies, itself being absent.[52] The third substance, therefore, may be said to mediate between the two commodities, serving as a relative measure that allows translating the value of one into the value of another. The contradictions can be overcome only when we consider *value* in a Spinozist manner, that is, as a phenomenon that manifests itself in the form of different, incommensurable attributes. When the same commodity is exchange-value and use-value for seller and buyer, respectively, then this is so because *value*, as a unitary phenomenon, may express itself in different, even and necessarily contradictory terms. Exchange, rather than being a thing, then is a living process, and value itself is a phenomenon encompassing sociological and temporal dimensions. (Exchange-) value is the suprasensible dimension of the sensible commodity, which, though suprasensible, materializes in something else, another commodity or money.[53] This relation is of the same kind as that depicted in Fragment 5.1.2, where the sound-word, qua sociological phenomenon, belongs to speaker and recipient simultaneously and, therefore, may manifest itself in the form of different 'meanings'. Whereas in constructivist theories, 'meaning' is something ideal, it is something concrete in the Spinozist-Marxian approach.[54] Thus, the (exchange-) value simply reflects the relations between things and between people now attributed to the things themselves.

Instead of taking recourse to an individualistic take, we may ask a sociological question: 'What social phenomenon is observed?' and 'Through what kind of *joint*, social work is the phenomenon produced in ways so that it is recognizable?' In this take, the phenomenon no longer is a function of *these* participants: *they* are then but the staff that perform a kind of recognizable concerted work and produce a recognizable social phenomenon that other people, finding themselves in other configurations, also produce.

In the preceding event fragment, we observe a conversation; and this one social phenomenon has different parts.[55] The parts are parts of a whole, and, therefore, they cannot be like independent elements that somehow come or are brought together to form the whole. For example, Mrs. T. can be seen to be a teacher by the way in which the event as a whole unfolds; if we had followed her around in a supermarket, we would have seen a grocery shopper rather than a teacher. We may not have had any clues about the fact that she also is a teacher. In fact, she is a university professor, and this professorial aspect of her life does not show through in

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[52] Karl Marx and Friedrich Engels, *Werke Band 23: Das Kapital: Kritik der politischen Ökonomie Erster Band Buch I: Der Produktionsprozeß des Kapitals* (Berlin: Dietz, 1962), 51; and Evald V. Il'enkov, *The Dialectics of the Abstract and the Concrete in Marx's Capital* (Moscow: Progress Publishers, 1982), 269.

[53] Marx and Engels, *Das Kapital*, 85–98.

[54] See Evald V. Ilyenkov, 'Dialectics of the Ideal', *Historical Materialism 20* no. 2 (2012): 149–193.

[55] For readers to get a feel for this kind of analysis, it would be useful and instructive to do what I describe elsewhere: Analyze a 'mystery' transcript about which you do not know anything, and which contains only arbitrary letters to indicate different speakers (e.g. 'A', 'B'). I show how an experienced analyst, in *every* case, reconstructs based on the words alone and without having any other information, actually comes to figure out what type of event produced the words, who the participants likely are, and so on. See Wolff-Michael Roth, *Rigor in Qualitative Data Analysis* (Rotterdam: Sense Publishers, 2015), chapters 2–6.

this event. It is the event that makes Mrs. T. as much as she contributes to making the event what it is. The same considerations pertain to the children, each of whom may be a ballerina, musician, or athlete, depending on where and when we follow them. That they are learners is contingent on the presence of a teacher, and that Mrs. T. is a teacher is contingent on the presence of one or more learners. Their mutual 'function' has to be revealed through analysis rather than taken from their institutional positions.

As an integral part of the event generally and the unfolding talk specifically, the function of Mrs. T. in this event is not external to the functions of other parts, the three girls, task, or materials (shoebox, plasticine shapes). To say she is a mediator means we need to abstract from the situation, reduce it to the individuals (Sylvia, Jane, Melissa) before we can say that she is somehow *between* the three girls. This is not to say that there is no effect of her presence – in the whole that we consider, she is an integral part, and, as such, shapes and is shaped by the event, the nature of which itself can be determined only in the end and after everything has been said and done. More specifically, we can state 'Mrs. T. mediated the girls' activity' in the classical take only if we know the outcome, that is, the fact that the three girls eventually arrive at a common model. If this outcome was not observed, investigators might state that the teacher failed to achieve the ('intended') mediation. Such an expression itself would point us to a social fact, for how could investigators have seen that the teacher 'failed' at achieving an 'intended' 'mediation' unless what she has done is recognized to be an integral part of a social situation in which one person assists others to come to an agreement. There is an apparent confusion between an institutional role and the actual function the participation of an individual has in a real, concretely experienced event that has happened and that now, after is has ended, may be grasped as a whole.

We therefore take what the fragment makes available to be part of an event in which an agreement is achieved – an assessment that we can make only *after the fact*. As such, our whole unit reaches, at a minimum, from the existence of difference to the achievement of agreement. In that whole, Mrs. T. is an *integral* part, not an external third. In and with her presence, the conversation takes a particular direction, and Mrs. T. is internal rather than external to the communicative exchange. If there is a particular effect – such as the initiation of an unfolding that eventually leads to a common statement with which all three girls agree – then that outcome itself is in part hers.

While Mrs. T. talks to the children, there is a common, intersubjective field, including (at a minimum) the three plasticine models, language, and the historically produced descriptions ('it is a cube', 'it is a rectangular prism'). To these aspects, the unfolding communicative exchange adds other descriptions, including 'there is only one object' and 'as a group you have to say, "it is …"'. Because all of these phenomena are integral part of the intersubjective field, they cannot be *between* other parts. These parts are part of all other parts, not in the least of Mrs. T., Sylvia, Jane, and Melissa. As Spinoza points out, things external to each other *cannot* affect one another; the only way in which things relate is when they are parts of the same substance. These parts are outside other parts only when we re-

gard the *one* substance in extensional terms, which means, if we reduce substance to its attribute *Extension*. But as soon as we reason in terms of extension, then the unity of the one substance has been lost because we reason in terms of a one-sided expression rather than in terms of an expression that reflects all attributes (manifestations).

For the same reasons, none of the objects can be thought of as a mediating artifact or material. Each entity – the shoebox and its content, the three models individually and as ensemble, the sound-words – constitute the world common to the interlocutors or they bear no relevance to *this* situation and event. (Recall also Feuerbach's contention of something common or as not having any relevance whatsoever.) We can state any properties in an abstract and abstracting sense. But if such properties have no bearing on the pragmatic unfolding of the event, then they are completely irrelevant to understanding why and how the event unfolds in the way it actually does.

Finally, to achieve anything at all that is a recognizable and therefore social phenomenon – such as an 'instruction to come to an agreement' or 'an agreement' – requires intersubjectivity. This is so because the phenomenon is produced in a *concerted* endeavor. In fact, the recognition of something as a 'disagreement' requires intersubjectivity. 'Negotiation' requires recognition that participants initially disagree and recognition that agreement has been achieved and in which way agreement manifests itself in whatever is said or seen. No attribution of a mediating role is possible unless there is also a recognition that whatever follows somehow is related to what the teacher has said before, which requires the recognition that whatever has been said also has had an effect and, thus, requires intersubjectivity. In the Spinozist-Marxian take of Vygotsky's last period of work, there could not be intersubjectivity based on intrasubjectivity, inherently thought as something that distinguishes between individuals. This is so because '*one substance cannot be produced by another substance*'.[56] For intrasubjectivity to produce intersubjectivity, they have to be manifestations of the same, in which case, intrasubjectivity inherently is intersubjectivity – expressed as the unity/identity of differences. Thus, from the Spinozist-Marxian position, it makes no sense to state that there is a mediating agent, the teacher with her subjectivity, standing as a third subjectivity between but bridging the separate intrasubjectivities of the girls.

From Mediation to the Intra-Intersubjective Field

Theorizing in mathematics education often begins with taking the individual as the unit of analysis; the social, then, is a result of 'social construction' of collective entities. This view, however, was already subject to Vygotsky's critique when he notes how psychologists 'assume that there is a special individual psyche and that from the interaction of individual psyches or psychologies there arises a collective

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[56] Baruch Spinoza, 'Ethics', in *Complete Works* (Indianapolis: Hackett Publishing, 2002), 219.

psyche or psychology common to all individuals'; he concludes that 'social psychology is regarded as the psychology of a collective individual in the same way that a crowd is made up of single individuals'.[57] This chapter shows why we ought to approach social situations in terms of intersubjectivity rather than intrasubjectivity. Not intersubjectivity but intrasubjectivity is the problem. This position arises directly out of the Spinozist take that Vygotsky was beginning to take near the end of his life. Because the thinking body takes the shape of the world it inhabits, there is nothing inside that is not already outside; the converse is also true, as Johann Wolfgang von Goethe stated, 'There is nothing outside us that is not simultaneously within us'[58]; and, with respect to the relationship between Self and Other, the French poet Arthur Rimbaud wrote, 'JE est un autre [I is another]'[59]. Vygotsky was later to note that the word is impossible for one person but a possibility for two, a phrase he attributes to the Spinozist materialist philosopher Ludwig Feuerbach, who suggested, 'What is absolutely impossible for one person, is possible for two' and 'only that exists what is for me and the Other simultaneously, wherein I and the Other agree, what is not only mine [mein] – what is general [allgemein]'.[60] In more recent years, both philosophers and psychologists have acknowledged that the Self is Another, leading to 'the "Other within" of psychology'.[61] In all of these writings and experiences, the starting point for thinking about who we are and what we think, believe, and feel is outside, the public and therefore common field that we inherently share with all others. We are always born into a world that already is populated with others and things and that we find, as anyone else, already there and affecting us. We might therefore say aphoristically: *The 'inter' always already is there. It is to human projects what to fish is the water in which they swim, live, eat, and get eaten.* That *inter* takes on specific significance because it both shapes and is shaped by human relations, which gives a special coloring to the idea that the human comprehension of the world and of other human beings is dramatic in nature[62], a realization that had become dear to Vygotsky.

Rather than taking intersubjectivity to be a problem, we might therefore start with the recognition that there is joint labor in some form of productive activity – a research project on the analysis of data and graphs or a mathematics task that constitutes an integral part of schooling. Participants do whatever labor is required to move the current task toward completion, whether the endpoint is clearly established beforehand or, as inherently in schooling, where participants find the goal of the task together with completing the task successfully. In accepting activity as the

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[57] Lev S. Vygotsky, *The Psychology of Art* (Cambridge: MIT Press, 1971), 14.

[58] Johann Wolfgang von Goethe, in *Gespräche mit Goethe in den letzten Jahren seines Lebens 1823–1832* by Johan Peter Eckermann (Leipzig: F. A. Brockhaus, 1836), 331.

[59] Arthur Rimbaud, *Œuvres complètes* (Paris: Gallimard, 1954), 252.

[60] Ludwig Feuerbach, *Sämmtliche Werke Band 2* (Leipzig: Otto Wigand, 1846), 283 and 308.

[61] Paul Ricœur, *Soi-même comme un autre* (Paris: Éditions du Seuil, 1990); and Feliks T. Mikhailov, 'The "Other Within" for the Psychologist', *Journal of Russian and East European Psychology* 39 no. 1 (2001): 6–31.

[62] Georges Politzer, 'Les fondements de la psychologie: psychologie mythologique et psychologie scientifique', *La Revue de la Psychologie Concrète* 1 (1929), 26.

unit of analysis, investigators can then begin with the analysis of labor in a specific social-material field populated by other people, tools, objects, and sound-words (see Fig. 5.4). All these entities are public, accessible to all, and language, while just as concrete as sound, also functions to produce the accented visible. But the accented visible is produced not by language alone. We see how a certain accented visible is produced in aligning the three plasticine models that make them the perceptual focus all the while the accompanying talk makes them the semantic focus. The visible changes when two of the objects are 'grouped' when they are moved closer to each other with a relatively larger distance to the third (Melissa's 'cube'); simultaneously, the grouping is accented in the opposition of 'this is a cube and these are rectangular prisms'. It is precisely in this way 'how a *semantic field – the accented visible*, different from the visible, is constructed' thereby becoming 'a place where something is made manifest and something, on the contrary, is covered over, annulled, so to speak'.[63] Both the perceptually visible and its accented nature are available in the public to anyone caring to witness and without having to go into the mind of any individual. It is there, present, and, therefore, does not have to be *re*presented. That common, *intersubjective* field is the condition for anything individual and the Self; and that 'real–ideal intersubjective field of their common "*co-spirituality*," the universal forms of their shared culture, both the culture of practical intellect and the culture of mind and spirit, are continuously created and perfected by these same behavioral acts'.[64] Anything pertinent to an event, including tools and language, is inconceivable outside common labor and common activity, which establishes a common field of purpose and sense. In the movement toward the ultimate production, a contradiction is now accented such that it can be, and has to be, actively resolved as part of the overall movement. Any individual form of consciousness also becomes part of the common field in and through the performance of language, which, in turn, is (historical) consciousness – literally, *common* knowing – externalized for the other.

Unlike (radical) constructivists, cited at the beginning of this chapter, Spinozism in psychology leads us to recognize that outside the common fields of experiences, the child has nothing and owns nothing: 'The child, born a human being, would remain forever without his own "Ego" if he were himself not one of the charged "items" in a field common to all, stuffed with the affective purposes of people's dealings with one another and continuously recharged by them'.[65] In other words, we might say with the Spinozist Marxian philosopher Il'enkov, 'the mind, from the beginning to the end, is a function and derivative of the *external action of the organism* – that is, of its movements in an external space filled with objects'.[66] This idea becomes especially salient in the education of deaf-blind children, who perceive neither written words nor spoken sounds and, thus, cannot ever 'make meaning' in the sense that this expression is used. Yet some such children end up be-

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[63] Boris D. El'konin, *Vvedenie v psixologiju razvitija* (Moscow: Trivola, 1994), 23.

[64] Mikhailov, 'The "Other Within"', 20.

[65] Mikhailov, 'The "Other Within"', 27.

[66] Evald V. Ilyenkov, 'A Contribution to a Conversation about Meshcheriakov', *Journal of Russian and East European Psychology* 45 no. 4 (2007), 88.

coming university professors. They do so because the real life of language is the language of real life, and these children equally learn a language (which, eventually, may be Braille) because they already know the language of life.

Near the end of his life while oriented toward Spinozism for working out a Marxist psychology, Vygotsky was engaged in a 'resolute and straightforward search for the original, inner, primal core of the mental, his search for the one single Principle of intellect and affect, of higher and lower forms of behavior, of nature and culture in the life activity of individuals of the species *Homo sapiens*, as they pursue that activity among the bare objects of existence'.[67] Unfortunately, Vygotsky never got to complete this work, immediately preceding his death likening himself to Moses who had seen the Promised Land without being allowed to set foot on it. The foundation of the unity of the thinking body lies precisely in the common field, which includes the speech field, no less material but special, as an integral part. It is generally accepted that E. Il'enkov developed what can be seen as an elaboration of fundamental concepts in Vygotsky's later thinking. Thus, 'if we enrich Vygotsky's ideas with Il'enkov's basic postulates, modem psychology and pedagogy will take a considerable step forward in study of the genesis and development of consciousness of the individual subject of activity'.[68] It is in this spirit and focusing on mathematics education that the present book was written.

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[67] Mikhailov, 'The "Other Within"', 19.
[68] Davydov, 'Reform', 92.

6

Pereživanie – Experiencing Mathematics

In a lecture from the last year of his life, L. S. Vygotsky articulated a unit of analysis [edinica] that does not reduce knowing and learning to the individual: *pereživanie*, often transliterated as 'experience', sometimes, but incorrectly, as 'emotional experience'. *Pereživanie* is an *event* in which we find the unity/identity [edinstvo] of person and environment; it always has practical, affective, and intellectual shading. Because *pereživanie* designates person and environment as parts of one and the same system, itself part of (and reflecting) the whole world and life, it is therefore a category consistent with the Spinozist-Marxian one-substance approach. This means that when investigating any performance situation in a mathematics classroom, we need to consider person-acting-in-environment as one single unit. This is a *transactional* understanding of events that the American philosopher John Dewey developed almost simultaneously with Vygotsky using the category *experience*. He also pointed out that 'experience' is a treacherous term, arising from the many different ways in which it is used, leading to 'continual confusion and unintentional misrepresentation'.[1] To avoid the confusion between the different uses of 'experience', the Russian term *pereživanie* is used here, though including the ideas of the American philosopher, who has articulated this theoretical category in ways that are almost identical to those of Vygotsky. The latter defined the category as '*a unit where, one the one hand, in an indivisible state, the environment is represented i.e. that which is being experienced* [pereživaetsja] – a pereživanie is always related to something which is found outside the person – *and on the other hand, what is represented is how I, myself, am experiencing* [pereživaju] *this,* i.e., all the personal characteristics and all the environmental characteristics are represented in pereživanie'.[2] As a result, *pereživanie* includes both personal and environmental characteristics (Fig. 6.1a), so that behavior never can be ex-

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[1] John Dewey and Arthur F. Bentley, 'Knowing and the Known' in *Useful Procedures of Inquiry* edited by Rollo Handy and E. C. Harwood (Great Barrington: Behavioral Research Council, 1999), 242.

[2] Lev S. Vygotskij, 'Lekcia četvertaja: Problema sredy v pedologii', in *Lekcii po pedologii* (Izhevsk,: Udmurdskij University, 2001), 75, original emphasis, underline added.

Fig. 6.1 a *Perežìvanie* is defined as the unity/identity of person and environment, which is not the objective environment but the environment as salient in the person's actions. **b** The definition immediately allows us to understand that when there are two persons in the same objective environment, the respective perežìvanie will be different

a.

person | environment

perežìvanie

b.
perežìvanie₁

person₁ ┊ environment ┊ person₂
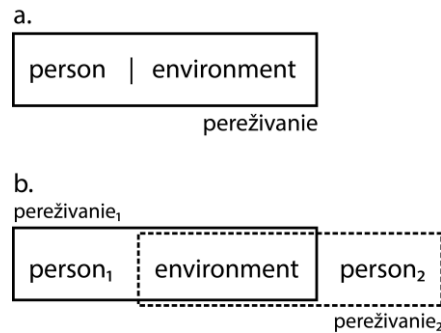
perežìvanie₂

plained purely in terms of the person (mental structures, conceptions) or purely in terms of the environment (e.g. social influences, 'peer pressure', and the likes) – as we can find this so often in the research literature. More so, *perežìvanie* does not just include those characteristics but in fact constitutes the 'unity/identity [edinstvo] of environmental and personal moments'.[3] In any performance situation, we cannot therefore understand the relevant personal dimensions without considering the environmental characteristics of the situation; and we cannot understand the relevant environmental dimensions without consideration of the personal characteristics. *Perežìvanie* is a *trans*actional unit, where each part is included in the definition of every other part of the whole situation.

Fig. 6.1a clearly exhibits the fundamental idea underlying *perežìvanie*, which is the {person | environment} unity/identity. When there are two persons, the physical environment may be thought of as being the same for the two (Fig. 6.1b) – a statement in which we neglect for the time being the fact that the other person also is a physical body and therefore part of the material environment. The diagram shows that given the same environment – which, in our context, could be 'the same mathematics curriculum' and 'the same mathematics teacher' – then, there are two *perežìvanija*. This also immediately applies that 'the same test' is not going to lead to the same *perežìvanie*, and, thus, 'the same test' will not be measuring the same at all. The diagram also gives shape to what is stated in chapter 5, which is the inherently shared nature of material things, including sound-words. A relation only exists if there is a shared world, otherwise there is no relation at all and, thus, there are no phenomena of the teaching | learning type.

From the History of *Perežìvanie*

Over the past 35 years, there has been an exponential increase of interest in the term *perežìvanie*, as indicated by the number of articles using the term in each five-year period (Fig. 6.2). This interest is also reflected in the fact that a recent special

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[3] Vygotskij, 'Lekcia četvertaja', 77.
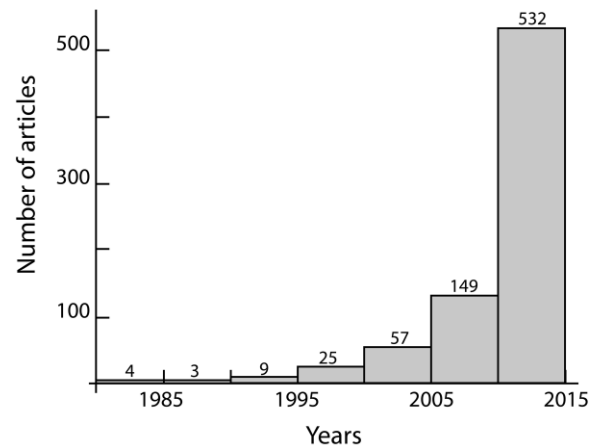
Fig. 6.2 Number of articles drawing on the concept pereživanie (perezhivanie) in each of the five-year periods from 1980 to 2015

issue of *Mind, Culture and Activity* was devoted to the notion, again showing that there are quite different takes on the category, making it the name for different concepts. In this book, *pereživanie* is taken as a category – and, equivalently, a unit of analysis – that extends over the person–environment relation and the reflection of this relation. It thus takes into account the 'identity of nature and man' and that 'the production of ideas, of conceptions, of consciousness, is at first directly inter-woven with the material activity and the material intercourse of men – the lan-guage of real life'.[4] Most importantly, it includes the fact that there is not just a relation of person and environment in the way this exists for most other organisms – in the way described in theoretical biology, as organism–Umwelt relation – but that the relationship exists *as* a relationship: that it exists for the person. This dou-bling of the relation is particular to humans, whereas 'the animal does not "*relate*" itself to anything, it does not "*relate*" itself at all. For the animal its relation to oth-ers does not exist as a relation'.[5] It is for this reason that the problem of conscious-ness becomes of primordial importance to Vygotsky during his final, Spinozist period. 'Consciousness is at first, of course, merely consciousness concerning the *immediate* sensuous environment and consciousness of the limited connection with other persons and things outside the individual who is growing self-conscious'.[6]

An important aspect of *pereživanie* is that 'consciousness' does not merely refer to the intellectual; instead, *pereživanie* also includes the practical and affective reflection of the relationship with the physical and social (no less material) envi-ronment. Consciousness replaces the instinct of the animal or, as K. Marx and F. Engels point out, consciousness is in humans a conscious instinct. Vygotsky direct-

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[4] Karl Marx, and Friedrich Engels, *Werke Band 3* (Berlin: Dietz, 1978), 30, 26.
[5] Marx and Engels, *Werke Band 3,* 30.
[6] Marx and Engels, *Werke Band 3,* 31.

ly refers to Marx when he states that the doubling of experience necessarily arises from human labor. Thus, paraphrasing Marx, he notes that 'such *doubled experience* [opyta] allows man to develop active forms of adaptation which the animal does not have'.[7] This doubling works such that 'being conscious of one's experiences [pereživanij] means nothing more than having them as an object (a stimulus) for other experiences. Consciousness is the pereživanie pereživanij [experience of experiences] in precisely the same way as *pereživanija* is simply the pereživanija of objects'.[8] Of utmost importance is the fact that that with the consciousness of *pereživanie*, an affective-valuative dimension enters our considerations so that acting/thinking/talking no longer is indifferent, thinking is no longer for itself but for changing a world co-inhabited with others. Thus, 'the active *pereživanie* of *pereživanie*, the active thinking of a thought, means not being absolutely indifferent to it, means an affirming of it in an emotional-volitional manner' so that 'the *pereživanie* of *pereživanie* and the emotional-volitional tone can gain their unity only within the unity of culture; outside they are fortuitous'.[9]

Attention to that double nature of *pereživanie* already can be found in one of Vygotsky's notebooks dating to 1926, where he writes on the origin of subjectivity: 'my body, for me, enters into two lines of *pereživanie*: inner-body and outer-object-oriented *pereživanie* (it is I and a body among bodies); this leads to the idea of something that is I, but is not my body'.[10] Later, in 1932, he would write in his notebook that the life of consciousness is unlike organic life, which 'places consciousness outside org[a]n[i]c life'.[11] Yet in the lecture on the role of the environment where the category *pereživanie* is articulated to the most extent, we still observe remnants of the old Vygotsky, focusing on the individual and attributing to it characteristics rather than orienting us to the *real* relations of people with each other and the environment. In that situation, the words are shared, are reality for two, and what matters is how word-use shapes the unfolding and unfinished event, which can be grasped only after the fact when there is a sense of completion.

Pereživanie is not something we ought to search for internal to the person, because it really designates the person and environment as parts of one and the same system, itself part of the whole world, Nature, and Life, which constitutes one substance. This is apparent in the example Vygotsky provides: three children who live with the same abusive mother. Objectively, notes Vygotsky, the environment is the

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[7] Lev S. Vygotsky, 'Consciousness' in *The Collected Works of L. S. Vygotsky Vol. 3: Problems of the Theory and History of Psychology* (New York: Springer, 1997), 68 (Russian ed., 85).

[8] Vygotsky, 'Methods', in *The Collected Works of L. S. Vygotsky Vol. 3: Problems of the Theory and History of Psychology* (New York: Springer, 1997), 41 (Russian ed., 50).

[9] Bakhtin, *Philosophy of the Act* (Austin: University of Texas Press, 1993), 35 and 36; also, '<K filosofii postupka>', in *Sobranie socinenij t.1* (Moscow: Izdatel'stvo russkie slovari jazyki slavjanskoj kul'tury, 2003), 35.

[10] Ekaterina Iu. Zavershneva, '"The Key to Human Psychology": Commentary on L. S. Vygotsky's Notebook from the Zakharino Hospital (1926)', *Journal of Russian and East European Psychology* 50 no. 4 (2012), 32.

[11] Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive: New Findings – Notebooks, Notes, and Scientific Journals of L. S. Vygotsky (1912–1934)', *Journal of Russian and East European Psychology* 48 no. 1 (2010), 54.

same, as shown in Fig. 6.1b for two individuals. The three children's *preživanija*, however, are different because the same physical-social environment is associated with different doubled *preživanija*. That is, there is both a relation and that relation exists *as* a relation differently for each of the three children.

In considering the relationship between a human being and its outside world, we usefully distinguish between a view on the world from without, her *environment*, and a view from within, her *horizon*.[12] Thus, although the three children lived in the *same* material-social environment (with respect to the mother), their horizons *differed* substantially.[13] To understand the behavior and expressions of each child, however, we need to understand the horizons rather than the objective structures of the given world. My participation in everyday life is circumscribed by the outside world, which constitutes the '*horizon* of my active, act-performing consciousness', so that 'it is only in cognitive, ethical, and practico-instrumental categories that I can … orient myself in this world as in an event and introduce a certain order into its composition with respect to objects'.[14] These objects do not exist abstractly and indifferently but '*stand over against* me as the objects of my own cognitive-ethical directedness in living my life'.[15]

From the perspective of method, the facts that (a) the second form of *preživanie*, the preživanie preživanij [experience of experience], is a *reflection* of the simple material relation and (b) horizon is a view from within activity do not mean that we have to get into the minds of individual participants. Instead, we can draw on the fact that participants themselves make available *for* and *to* each other the relevant aspects of the situation, the order of things, and, simultaneously, make available *for* and *to* one another the methods of producing this order. In other words, to understand human behaviour, we do not need to make inferences about the individual minds but need to attend to whatever is transactionally relevant: Does a participant take up what another one has said or done? Is it topically relevant to other members to the setting? In any setting, participants are accountable for their actions and therefore employ methods that are 'uniquely adequate to phenomena of order, to things, to the unavoidable objectivity of social facts in workplace specific phenomenal field properties in and as of their generality; and … that therein methods be congregationally exhibited and congregationally witnessable in embodied workplace-specific, discipline-specific details of making and describing *things*'.[16]

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[12] See Mikhail M. Bakhtin, 'Author and Hero in Aesthetic Activity', in *Art and Answerability* (Austin: University of Texas Press, 1990), 97–99.

[13] Newman et al. suggest that the same environment will lead different groups of children to do a different task, though from an instructional perspective there is only one and the same task in different contexts. See, Denis Newman, Peg Griffin, and Michael Cole, *The Construction Zone: Working for Cognitive Change in School* (Cambridge: Cambridge University Press, 1989), 22 and 122.

[14] Bakhtin, 'Author and Hero', 97–98.

[15] Bakhtin, 'Author and Hero', 98.

[16] Harold Garfinkel, *Ethnomethodology's Program: Working out Durkheim's Aphorism* (Lanham: Rowman & Littlefield, 2002), 102–103.

The Person–Environment Unity/Identity

The Russian *pereživanie* – as the English ex*per*ience, the French ex*pér*ience or the German equivalent Er*fahr*ung – is related to travel, traversal, peril, risk, and change. The etymological root *per(e)-* denotes the verbs to try, dare, and risk, put oneself in danger; as such, it also made it into such words as ex*per*iment (Greek *peírama,* experiment) and *per*ilous. The root brought into modern languages the sense of carrying over, bringing, going over, faring. As such, the root refers to transition, going through and undergoing something. A third sense pertains to going over, leading toward (e.g. 'far', 'afar'), further, later, forward (prone, pronate) toward the end, a limit (perimeter). That is, *pereživanie is both a process and that which, at its limit, the process gives rise to*: That of which *pereživanie* is a *pereživanie* is an event. We do not control events: we live through and undergo them (as witnesses, patients) *as much as* we are agential subjects therein. *Pereživanie* designates the person and environment as parts of one and the same system, itself part of Nature (Life) that constitutes, in the Spinozist-Marxian take, but one substance with the attributes Extension and Thought.

There are two dimensions of *pereživanie* that tend to be under-theorized in the research literature: its spatially distributed and its temporal nature. That is, *pereživanie* is characterized by a double dehiscence: spread out over space and spread out over time.[17] The two dimensions of experience, the spatially distributed and temporal, are inseparable: 'the two principles of continuity and interaction are not separate from each other. They intercept and unite. They are, so to speak, the longitudinal and lateral aspects of experience'.[18]

First, *pereživanie* is a category, an indivisible unit and therefore identity of environment and person (Fig. 6.2). Vygotsky explains, in '*pereživanie we are always dealing with an irreducible unity of characteristics of personality and characteristics of situation*'; but 'it is not any of the moments in themselves … which determines how they will influence the future course of his development, but the same moments refracted through the prism of the child's *pereživanie*'.[19] Taking into account the environment in understanding behavior is deeply Spinozist-Marxian, for, as suggested in chapter 1, what we call 'thinking' cannot be understood in itself because it is part of a series of events the beginnings and endings of which are situated outside of the thinking body. Therefore, 'what we call internal *pereživanijami* [experiences] is … taking place on the surface of the system of consciousness' but 'its study cannot be separated from the investigation of practical activity'.[20] In this way, the nature of *pereživanie* is distributed; it always consti-

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[17] In fact, the experience of experience is at the origin of time, because we can experience an experience as a whole only when it is finished. Then, to be experienced, it has to be made present again, that is, it has to be represented. That distance between experience and its grasp as experience by means of a representation is a temporal one.

[18] John Dewey, *Later Works Vol. 13* (Carbondale: Southern Illinois University Press, 2008), 25.

[19] Vygotskij, ' Problema sredy', 73.

[20] Alexei N. Leont'ev, 'Dejatel'nost'. Soznanie. Ličnost'', in *Izbrannye psixhologičeskie proizvedenija tom 2* (Moscow: Pedagogika, 1983), 185.

tutes a unity/identity of characteristics of both person and environment poles of the relation, which is folding back over itself so that it becomes presented again in *pereživanie*. In this, Vygotsky's *pereživanie* shares an aspect with the Deweyan *experience*, as long as that term is 'held strictly to a single definite use: that, namely, of calling attention to the fact that *Existence* has organism and environment as its aspects, and cannot be identified with either as an independent isolate'.[21] Nevertheless, Dewey and Bentley shift their attention to practice and *behavior*, which they ask to be used in place of the ambiguous term experience.

Second, *pereživanie* is a category of change and therefore not only changing the person but changing itself as well. Dewey therefore suggests relative to his concept that 'experiencing like breathing is a rhythm of intakings and outgivings'.[22] *Pereživanie* then also constitutes a movement: it is an event. It is a process of *trans*action, involving external, objective and internal conditions. In this it is like the Deweyan experience, which 'is always what it is because of a transaction taking place between an individual and what, at the time, constitutes his environment'.[23] *Pereživanie*, as the Deweyan experience taken in the strong sense, thus, spreads over time and, therefore, captures the whole *person-in-environment* while also undergoing change. In fact, we might add verbs so that the expression becomes something like: *person-acting/thinking/feeling…-in-environment. Pereživanie* therefore cannot be characterized in terms of (mental) structures, for the minimum analytic unit *is* change that, if anything, could manifest itself in terms of a plurality of structures. It is impossible 'to divide in a vital experience the practical, emotional, and intellectual from each other and to set the properties of one over against the characteristics of the others'.[24] But there is a second, cultural-historical dimension to experience in the way Dewey understands it: If we were to destroy the external conditions of our experience, that is, humanity would have to start again where it had been some time in the past. *Pereživanie* is the locus and result of development, and, because '*man is a social creature … without interaction with society, he will never develop in himself any of the qualities and attributes which have developed as a result of the methodical evolution of all mankind*'.[25] Dewey, too, rejects the idea that experience [pereživanie] is something that can be reduced to persons and their minds (in the way this would be theorized in constructivist terms).

The English translation of the posthumously published text of the lecture on the role of the environment uses the term *pereživanie* in different ways, sometimes in the way in which English speakers use 'experience' or 'feeling' (as in 'how I am experiencing this') and at other times clearly including the relationship to something that occurs outside of it (so that the environment is represented). The contradiction is apparent in this statement: 'So, *in a* pereživanie *we are always dealing with an indivisible unity/identity of personal characteristics and situational char-*

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[21] Dewey and Bentley, 'Knowing and the Known', 193.

[22] John Dewey, *Later Works Vol. 10* (Carbondale: Southern Illinois University Press, 2008), 62.

[23] Dewey, *Later Works Vol. 13*, 25.

[24] Dewey, *Later Works Vol. 10,* 61.

[25] Vygotskij, 'Lekcia četvertaja', 89, original emphasis.

acteristics, which are represented in the pereživanie'.[26] *Pereživanie,* therefore, is a form of double description including, on the one hand, the indivisible person–environment unity/identity *and* the refraction of this unity/identity in that same unit. This is consistent with other monist approaches, where it has been recognized that 'to produce news of difference, i.e., *information*, there must be two entities (real or imagined) such that the difference between them can be immanent in their mutual relationship'.[27] This is important, for example, for the emergence of new forms (Vygotsky uses the term neoformation). One such system in the cultural-historical approach is the organism–environment, which allows us to understand the emergence of new biological and cultural forms (e.g., higher psychological functions) as the result from coincident contradictions that appear at the two poles (environment, organism). Importantly, the difference must be represented inside the system so that it bears on the relation. *Pereživanie* therefore integrates cultural-historical and affective dimensions of specifically human life. Thus, Bakhtin suggests that 'the experiencing of an experience [*pereživanie pereživanija*] and the emotional-volitional tone can gain their unity only within the unity of culture; outside that unity they are fortuitous'.[28]

A relational system has to be theorized differently than in terms of the traditional cause–effect relations that also characterize constructivist and other approaches. This is so because person and environment are not independent. Instead, what are relevant environment characteristics in performance depends on the person, and what are relevant person characteristics depends on the environment. When considering the relation of two individuals, we therefore have to expand our initial model (Fig. 6.1) to include the mutual other in the respective pereživanie. For example, for person$_1$, the physical environment and person$_2$ are part of what we might refer to as the physical-social environment1, leading us to a person$_1$ | (physical environment + person$_2$) unit (Fig. 6.3a). For the second person, the situation is accordingly (Fig. 6.3b). When analyzing a classroom event involving two individuals, we have to put these two models together giving rise to an irreducible unit of overlapping and mutually constitutive *pereživanija* (Fig. 6.3c). There is therefore a co-dependence of persons, environments, and *pereživanija*. Because of this co-dependence, it is more appropriate to speak of a *trans*action rather than *inter*action, a notion that presupposes the independence of the person and environment.

Consider, for example, a system consisting of a rabbit–dog relation. We may then ask, 'What redundancies must exist in this system so that this part of the system will be able to chase that part? And, perhaps, be unable to *not* chase it?'[29] The consequence is that all redundancies are relational kind, making it impossible to attribute cause and affect in the way that natural sciences do. Did the dog chase and thereby get the rabbit to run? Or did the rabbit run leading the dog to chase? In the hand-light switch relation, it is the hand that switches on the light, but the

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[26] Lev S. Vygotsky, 'The Problem of the Environment', in *The Vygotsky Reader,* eds. René van der Veer and Jaan Valsiner (Oxford: Blackwell, 1994), 342.
[27] Gregory Bateson, *Mind and Nature: A Necessary Unity* (New York: E. P. Dutton, 1979), 68.
[28] Bakhtin, *Philosophy*, 35; also, '<K filosofii postupka>', 35.
[29] Bateson, *Mind and Nature*, 136.

Fig. 6.3 a. The *pereživanie* of person₁ includes the material environment and person₂. b. The *pereživanie* of person₂ includes the material environment and person₁. . When analyzing events involving two individuals – e.g. two students, a student and a teacher, or two scientists – there are transactional relations involving people, environments, and *pereživanija*.
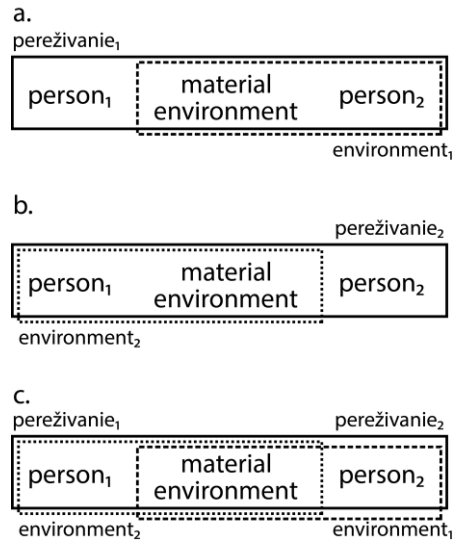
switch enables the hand to act. It is all too easy to consider the hunter, gun, and rabbit as a system composed of separate parts that *inter*act; and it is that very description that leads us to identify cause independently from the effect. However, when we take into account the history of hunting, including the gestural and vocal designation and later naming of the thing hunted, and when we take into account the global movement of hunter and hunted, including all levels of description and naming, then hunter and hunted no longer are independent but interpenetrating parts of the same system – Nature – describable only by a *trans*actional relation. Here, the action of one can no longer be separated from the action of another.

An Exemplifying Lesson Fragment

To exemplify the use and power of the category *pereživanie*, we turn to the episode also featuring in chapters 2 and 5 in which three girls are completing the task of modeling a mystery object hidden in a shoebox. We take a closer look at an early fragment of the episode, where, after having begun to produce the plasticine models, Jane, Melissa, and Sylvia begin talking about the object. When the fragment opens, Sylvia is reaching into the box saying very slowly that she feels it. The reply contains, among others, a conjecture concerning the nature of the object, 'a cube', and it is 'huge'. When the subsequent replies contest this description, a further reply provides a description of – which also serves as instruction for – how to go about ascertaining the shape by measuring its dimensions in three directions (turns 09–10) by feeling around (turn 11). The subsequent reply makes apparent

that the speaker is following the instructions, but, whereas feeling a square-shaped top, also is feeling a rectangular side (turn 15).

Fragment 6.1
01 S: <i fee::l i:t>
  a   (0.4)
02 M: f*eel*it=eh?
  a   (1.0) ((*J nods*))
  b   i have felt that cu:be
  c   (1.5)
  d   H*U*::::::=gë.
03   (1.6) ((*J grimaces, doubtfully?*))
04 S: *no* its n*ot* a cube ((*shakes head while rH in box*))
05   (1.4)
06 J: i didnt feel a cube.
07 S: me *ei:*ther.
08   (2.8)
09 M: *i* d*id*.
  a   (0.6)
  b   *I::* (0.2) i checked (0.5) the sides like that.
10   (2.5) ((*J watches S turn and 'measure' the cube, as in figure below*))

11 M: ((*J puts lH into box*)) you should feel around.
12   (1.6)
13   t fee:l it
14   (2.9)
15 J: ((*puts lH into box*)) if i feel the t*op* (0.4) it seems like its square but if i feel the side it seems like rectangle
  a   (1.0)
16 M: i dont f*eel* a rectangle.

In this fragment, we observe the multiple dimensions of relations between any one person and the environment, which includes the other persons (see Fig. 6.3c). There is, for example, a material relation between Jane and the mystery object that she is feeling over as the event unfolds and constituting an irreducible part thereof. That relation is reflected in consciousness, that is, the sense in which the object gives itself to her touch; and that tactile form of consciousness comes to be manifested in the model she is building. That model manifests consciousness for the other, here Melissa and Sylvia, as much as it manifests consciousness for herself.

Language, materializing as speech, here is the consciousness that in this practice exists for other people, and, as Vygotsky noted following Marx, therefore also for the speaker. That is, we do not have to speculate about mental structures and hidden aspects of intrasubjectivity. Instead, following Vygotsky's Marxist understanding, consciousness reveals itself for *all* participants and for the analyst in the words at hand. These sensible sound-words are part of the material environment that the participants *share* (Fig. 6.3c). The same form of reasoning applies to material objects. That is, the very possibility to be a model of something else, here the mystery cube, also implies the possibility of being a model for someone else. In fact, the model here functions like language, being both an integral aspect of the event and reflecting the primary relation. The relation has doubled, captured in the category of the doubling of *pereživanie*.

There is also the relation to others, here Melissa specifically. Their relation to the mystery and model objects (nature) conditions the to each other; and, conversely, the relation to each other conditions their relation to the objects. *Pereživanie* pertains to this doubled relational aspect, which is also conditioned on the cultural history within which the very capacity of 'naming', by means of words or models come to stand for another segmentation of the material world. That naming becomes an integral part of the cultural world much like 'gentlemen' and 'women' or equivalent iconic signs marked on certain doors have become a constitutive part of our environments that orient, shape, and give sense to our behavior.

As in every other fragment in this book, any turn here is a reply to a preceding turn and to the event as a whole. Thus, for example, 'feel it, eh?' is the reply to 'I feel it', which itself is a reply to whatever has preceded. 'Feel it, eh?' is only the second part of a response. As seen in chapter 3 (Fig. 3.10), a *response* always includes an orientation to the other, an active reception, and a reply. The reply, which is shaped in the active transformation of what is ringing in the ear, therefore, cannot be understood independent of the active reception. Whatever is received has come from the *environment* and appears as the *horizon* within which the reply forms. In fact, hearing is better understood as a resonance phenomenon or as a relation of coupled oscillators – which, in the easiest of cases with dead objects lead to coupled differential equations that cannot be solved independently. Thus, whatever thinking occurs in, and is shaped by, the formation of the reply has its origin not within thought, not within the mind of Melissa, but in the environment, as reflected in her horizon. Whatever the horizon, it is part of what the actors in the situation make available for each other in and through the joint work of completing the task. Moreover, what that thinking brings about, its effect, can be found in the next reply, itself part of a response that includes the statement 'I have felt a cube, h*uge*': 'I didn't feel a cube'. That is, neither the beginning nor the ending of the thinking | speaking lies and can be found *within* Melissa, but exist in the related person–environment/horizon relation.

There is also more than a simple semantic dimension, for the changing intonation manifests an affective valuation. The relation to the environment always is practical, whether it is with the material environment (Melissa and her plasticine cube, Jane and the mystery object) or the real intercourse with other people. But
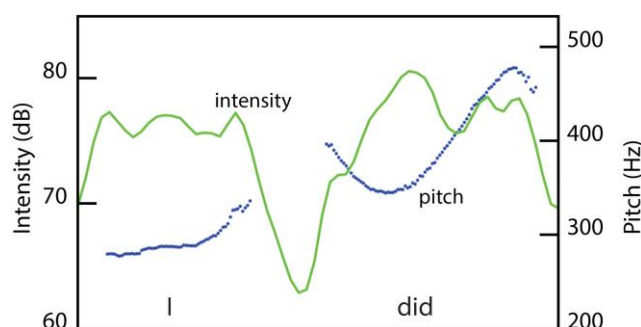
Fig. 6.4 Intonation and intensity of the sound-word sequence 'I did'

pereživanie encompasses more, as the practical person–environment relation also is refracted intellectually and affectively. *Pereživanie* has material-practical, intellectual, and affective manifestations. We observe further evidence for the affective-valuative dimensions in the changing facial expression of Jane, which we might gloss by means of the adjective 'doubtful' (turn 03), or by the emphases in '*I* d*id*' (turn 09, Fig. 6.4), which bring out the personal stake in the issue indexed here by means of the pronoun 'I' and the facticity of the feeling in the hand that yielded a cube. Nothing of this is contained in the semantics of the word and everything, as Vygotsky and Bakhtin repeat throughout their works, in the intonational qualities of the sound (resonance!). They do not just reflect something on the inside of the person but precisely the relation of person to environment in view of the current project as a whole.

A word such as 'cube' therefore does not simply designate an object, but, as sound-word and by means of intonation, manifests a valuative relation toward the object of talk. As such, that sound also affects another person, not intellectually, put precisely affectively (emotionally). In so doing, the word 'sets [the valuative attitude] in motion toward that which is yet-to-be-determined about it, turns it into a constituent moment of the living, ongoing event'.[30] Thus, especially the much louder than and emphatic '*huge*' manifests an affective appreciation of the situation. Moreover, there are many students acting in this way simultaneously, chiming in with each other, and thereby manifesting a collective affective orientation to the task. We can hear the teacher's voice radiating a matter-of-factness that emphasizes the mathematical-intellectual over the affective side of the activity. The spoken word, because of its materiality, is a manifestation of affect, for the recipients as much as for the speaker. (Vygotsky also emphasizes that in speaking, thought becomes itself.) As a result, in 'the active thinking of a thought', 'the active pereživanie of pereživanie … means not being absolutely indifferent to it, means an affirming of it in an emotional-volitional manner'.[31] The *pereživanie* of *pereživanie,* which *is* consciousness, obtains its unity/identity within culture.

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[30] Bakhtin, *Philosophy*, 32–33.
[31] Bakhtin, *Philosophy*, 34.

The need to take into account the surroundings is apparent when considering Jane's response, which begins with attending to the instruction received, the apparent exploration of the mystery object, and the verbal reply that accents the two shapes that can be felt: square on the top, rectangle on the side (turn 15). Again, the mystery object gives itself to the feeling hand, which therefore also receives the object, resulting in a particular sense that is related to the cultural-historical experiences of square and rectangular (non-square) surfaces. Even though all three girls feel out the same object – as ascertained through touch by the teacher, who reaches into the box, and accented by her in talk (chapter 5) – there are three different horizons. The 'same object' actually has felt differently. Because the mystery object gives itself – being available to all three – their *preživanija* are not 'mere' 'constructions'. The girls find themselves with feeling a cube or with feeling what the teacher calls a 'rectangular prism'.

The fragment makes explicit a temporal dimension of *preživanie*. To repeat, *preživanie* is not a thing but an event; there therefore is a temporal dimension to it. That is, we cannot speak of a person–environment relation as if there were a social fact but have to consider an event in the making where the environment continuously changes. For example, with each sound of 'I have felt that cube' (turn 02b), the physical environment is changing, thus the person–environment relation both in its simple and its doubled form. Thinking therefore also changes, continuously taking into account and reflecting the changing relation. Similarly, from the recipient side, the response does not come after a peer has said something. Any response, as shown in chapter 3, includes actively listening and replying. Instead, the respective responses begin with actively attending to what another is saying and end when she has finished her Saying in turn. It is not that any of the girls merely talks, but the talking reflects consciousness of the situation generally and the pertinence of the preceding statement specifically. When a factual statement enters the public sphere, the environment has changed; and when counterstatements follow, the environment not only changes again but now also features a contradiction that somehow seeks resolution. Two contradictory statements – 'it is a cube' and 'no it's not a cube' – cannot be true at the same time, especially not when there is an explicit requirement for the task outcome that demands but one, generally agreed-upon statement concerning the shape of the mystery object.

As soon as we choose *preživanie* as an analytic category, the whole cause–effect reasoning – used explicitly or implicitly in mathematics education research of qualitative as much as of quantitative kind – falls to its knees. This is so because, as shown in chapter 3, there is a mutually constitutive (social) relation between turns. 'It's not a cube' is a contradictory statement because there already is a statement with the opposite content ('I felt that cube'); and the nature of the initial statement, a factual one, is conditioned by the fact that there are other factual statements about the same object. 'You should feel around to feel it' is an invitation and instruction because there is an acceptance and a followed instruction, the nature of which is in turn conditioned on the former. *Preživanie* orients us to the unfolding event, produced and undergone, which implies that an agent *cannot* know the effect of her action and, therefore, in a strong sense cannot know what

she is doing while doing it. 'I didn't feel a cube' and 'me either' *contradict* 'I have felt that cube' because they are followed by a repeat assertion 'I did'. If, on the other hand, the two expressions had been followed by the reply 'you are joking' or 'you are ganging up on me', then their effect and impact on the unfolding event would have been different; that is, the exchange of them might have been glossed correspondingly. Once we consider the event as a whole, from its beginning to its ending, then nothing dear to other theories remains constant: persons, environment, statements, or objects are changing over the course of the event. These aspects are Spinozist through and through, for the philosopher was conceiving the one substance (Nature, Life) as self-moving; and Life is 'the force through which things persevere in their own being'.[32]

There is another important aspect to the temporal dimension, but now in reverse direction, paralleling the determination of the cause by the effect. In this episode, the children relate to each other, and, as shown in chapter 5, to their teacher. Some of these relations later show up as individual behavior. That is, if we observe in this episode a collective behavior that later is observable as individual behavior then the future already is present. Vygotsky describes this in what may appear to be a cryptic statement: 'everything "*possible to achieve at the end and as the result of the developmental process, is already available in the environment from the very beginning*"'.[33] The phenomenon is referred to as *prolepsis*. It is not, however, as if the teacher or others 'represent the future in the present'[34]; instead, as Vygotsky clearly points out, as shown in chapter 4, what *will be* an individual behavior (socially through and through) *will have been*, in every single case, a social relation (behavior) before.

When *perezǐvanie* becomes the object of another *perezǐvanie*, then events come to stand out of the stream of lived experience. For Dewey, from the stream of experiences '*an* experience' comes to stand out as a complete whole. It may then serve as an opportunity to be brought to bear, in consciousness, on future experiences. An opportunity to observe such a development arises when teachers ask students to report back to the class. Reporting back constitutes an invitation to summarize what has been standing out from an antecedent engagement. In the present lesson, such an opportunity arose when the teacher moved from group to group asking each to report back. Jane told the whole class what turns out to be a repetition of the statement she has made in Fragment 6.1, which is that the top feels like a square whereas the side feels like a rectangle. The teacher then articulates what will turn out to have functioned as an invitation to Melissa to say something about 'the other faces'. It is at that point that we observe an accepted (turn 07) articulation of the essence of an experience that has led her from the cube to the rectangular prism (turn 05). The two verbal indices 'that' are associated with finger and hand movements toward and around the rectangular prism, the first pointing to the rectangular side and the second with the caliber grip along the edges of the square

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[32] Baruch Spinoza, 'Ethics', in *Complete Works* (Indianapolis: Hackett Publishing, 2002), 218.

[33] Vygotsky, 'The Problem of the Environment', 347–348.

[34] Michael Cole, *Cultural Psychology: A Once and Future Discipline* (Cambridge: Harvard University Press, 1996), 184.

top. Following the interjection that marks acceptance ('okay'), there is what will have been an accepted (turn 08) rephrasing: 'so there's different types of faces' (turn 07).

Fragment 6.2
01 T: about the other faces what else do you feel m- melissa
02 M: um: (0.4) i fel'that (0.8) it ha- got it had a flat
          top and it doesnt m*a:d*()tch: ((*sideward
          glance*
03   (0.6)
04 T: what d*oes*nt match (0.2) whats d*iff*erent
05 M: that (0.3) s*i:z*e of ((*points, as left*)) (0.3) of
          that f*ac*e ((*caliper grip, as right*))
06   (1.0)
07 J: °okay° so theres different types of faces?
08 M: °yea°
…
24 M: um (0.5) <i f*elt*>
 a   (1.4)
 b   i felt the faces (0.3) tha:t ((*runs
          finger over top, square face of
          plasticine shape*)) (0.5) and=i felt
          th*is* part is actu'ly like that ((*rec-
          tangular side*)) (0.7) i felt th*a::t*
          ((*rectangular side*))
 c   (1.5) ((*sideward glance to the au-
          dience*))

 d   <and it felt sm*oo*:th> ((*runs finger
          down rectangular side*))
 e   (1.4)

 f   and (0.5) it was li()ke (0.2) fl*at*=like th*at*=
          ((*holds plasticine like a sandwich*)) instead'f
          li()ke a c*ub*e that is like (0.8) like (0.4) extra
          b*i:g*
25 T: ((*nods*)) it was like a flat cube
26 M: yea

    After two accepted invitations (Sylvia, Jane) to talk about the different types of faces felt, Melissa has another turn at articulating the preceding *pereživanie*. The

statement, acknowledged and glossed as 'it was like a flat cube' is oriented toward the rectangular sides much smaller in extension in the vertical direction than in the horizontal direction, which also constitutes the lengths of the square's edges.

It is precisely that what is emphatically articulated (turn 05), the size that has been the problem during the earlier part of the lesson when, after an extended period of feeling a cube, Melissa all of a sudden, with great astonishment, felt a rectangular prism in her hand. That same *pereživanie* comes to be articulated and accounted for here while the finger runs down the rectangular sides of the rectangular prism model, which are hardly higher than the width of a finger and small compared to the edges of the square top face. The difference of the extensions in the horizontal and vertical directions are iconically depicted when the model is held in a configuration, as if Melissa was holding a sandwich into which she would soon be biting (turn 24f). The concurrent words accent the visible flatness of the object compared to the size it would have had it been 'extra big' 'like a cube'. The lesson therefore gave space for a reflective account of an earlier *pereživanie*, making that one the object of the current *pereživanie* (related to providing an account). It is a turning point: whatever has remained in memory indexes the earlier being-there; it now is used to turn something into conscious being, making it an object of consciousness (i.e. Marx's *Bewußtsein*, consciousness, that is *bewußtes Sein*, conscious being).

This account, as all others, does not just make a factual statement but also constitutes the acceptance of an invitation to retell what has been done and learned. As such, it is a *social* rather than individual phenomenon. *Pereživanie*, as something that can be made present again, requires some form of 'technology' by means of which it is made to exist in the present, allowing *pereživanie* to be re-presented; and any such technology, because standing for something other than itself, inherently is technology for the other. In contrast to other ways to make the past exist in the presence, the plasticine has itself been a part of that past *pereživanie* to be made present again. It therefore has synechdochical function, a part that has come to stand for the whole of the preceding pereživanie. This is so not in the least because it allows performing an account *for* the Other. In the present instance, this Other is actively acknowledged in and by the sideward glances of the gaze toward the peers and one toward the teacher, a gaze otherwise entirely absorbed in the object. Those sideward glances also provide an opportunity to perceive (a) the effect of what has been said so far and (b) the evaluative part of the response on the part of the recipients.

In this account, the central object (the plasticine model) also has been part of the previous *pereživanie*. It functions as an anchor and support for a retrospective account in which it also figures as the main protagonist. Having a central part in the narrative, the object also shapes this narrative, which has to account of this object's part in the unfolding event recalled. Moreover, it now becomes an *integral* part of a communicative event, which is shaped by and shapes that part. This is so because it constitutes something visible and is part of accenting the visible. Being held in hand that enables the feat of making the past *pereživanie* return and become present. Concerning this point, one cultural historical study of the social psychology

of experience states, 'the past can only be brought into the present as a result of the most supreme efforts. One must journey into the "dark region" of one's own experience'.[35] Here, the model standing in for the mystery object, with its different faces, provides a structure around which the narrative of the different tactile events can unfold. Its 'flat top' can be contrasted – visually for others, visually and tactilely for herself – with the 'size of that face' that the audience can see. Likewise, the audience can see how the middle finger runs down the narrow face while it hears a description that 'it felt smooth'. The model affords being held like a sandwich, providing a visual ground for the description that 'it was flat like that', which contrasts 'a cube that is … extra big'.

In a strong sense, the account is not about Melissa but instead an account of a relation. A narrative inherently intertwines the protagonist and the plot, the relationships with things, phenomena, and other people. Only what can be accounted for in some such way is the *pereživanie* of *pereživanie*. Even the account of a dream is an account of a relationship with something that stands out, and, therefore, is part of a subject–object relation with the person. This is so because 'even when they sleep people maintain the use of speech to the extent that speech is an instrument of comprehension'.[36] Thus, even when the object of the *pereživanie of pereživanie* is something involving only us in the transition from deep sleep to a (partial or full) waking state, it is social through and through. Language is but one of the inherently societal (i.e. universal) ways in which consciousness – knowing together – is made accessible to the Other as much as to the Self.

Toward a Holistic Take on Mathematical Learning

Pereživanie – as Dewey's *experience* in a strong sense – being neither objective nor subjective, offers an opportunity of organizing our accounts of the relationship between person and environment. *Pereživanie* therefore is not subjective, because of its orientation to the material-social relation. It is also subjective in the sense that a person acts within the given horizon. In a study of the event of becoming aware, I describe how the forms and contents of thinking change once a person becomes conscious of something in the environment, that is, once the current horizon has changed.[37] To understand behavior, we therefore need to investigate the environment as refracted in the event, whether it involves several or only one person, rather than the objective environment as apparent to the investigator. *Pereživanie*, the unity/identity of person and environment, captures precisely this personally relevant aspect of the environment. But anything that stands out inherently can do so for others as for the Self. Vygotsky turned to an articulation of the category

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[35] David Middleton, and Steven D. Brown, *The Social Psychology of Experience: Studies in Remembering and Forgetting* (London: Sage, 2005), 139.

[36] Maurice Halbwachs, *On Collective Memory* (Chicago: University of Chicago Press, 1992), 44.

[37] Wolff-Michael Roth, '*Becoming Aware*: Toward a Post-constructivist Theory of Learning', *Learning: Research and Practice* 1 (2015): 38–50.

in the context of his '"global task [that] was to create a single theory of the psy-chic, specifically a general psychological theory of consciousness in which all pre-vious approaches would be 'sublated"'[38]; and in this task, consciousness was to function as a system of social relationships. It is a category that directs and forces us to go about studying human behavior in a relational manner. That is, 'when a name is wanted to emphasize the interconnectedness of all concerns, affairs, pur-suits, etc., and it is made clear that *experience* is used in that way, it may serve the purpose better than any word that is as yet available'.[39]

The relational approach embodied by the category of *pereživanie* especially comes through in the associated use of the term *trans*action to replace the notion of *inter*action, with its tendency to treat individuals as independently understandable 'elements' of relations with others and with things. It has been suggested to use the term *trans*action 'where systems of description and naming are employed to deal with aspects and phases of action without final attribution to "elements" or other presumptively detachable or independent "entities", "essences", or "realities", and without isolation of presumptively detachable "relations" from such detachable "elements"'.[40] Any report of human behavior, therefore, needs to ascertain that the 'organism-in-environment-as-a-whole' is taken into account. John Dewey and Ar-thur Bentley refer to the 'grosser abuses' associated with the term *inter*action and *inter*actional explanations. These include statements in which selves are said to interact with other selves and the physical environment and when the 'meaning' of a word is analyzed independent of its actual use and function in the real practical activities of the participants in the event under investigation. These authors also propose a way around the abuses of the term, suggesting another one that is fre-quently employed in this book especially in the context to emphasize the junction between the social and the individual: behavior. Thus, 'transactionally employed, the word "behavior" should do the work that experience" has sought to do in the past, and should do it free from the shifting, vague, and confused applications which have in the end come to make the latter word so often unserviceable'.[41]

In a narrower sense, this means that we need to account not for Melissa or Jane or Sylvia, but always for Melissa (Jane, Sylvie)-in-this-mathematics-lesson-given-this-task. In a more expanded sense – paralleling the expansion from cell as part of organism-as-a-whole to part of 'organism-in-environment-as-a-whole' – our ac-count needs to be expanded to one of Melissa in terms of Melissa-in-school, which itself is only a small part of Melissa-in-Society or Melissa-in-Life. We observe this move from a specific productive societal activity to the whole of a person and life in the category of personality, which Vygotsky and Alexei Leont'ev formulated in terms of Marx's Sixth Thesis on Feuerbach according to which 'the essence of *man* … in its reality, is the ensemble [sic] of societal relations'.[42] That is, in the defini-

[38] Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive (1912–1934): New Findings', *Journal of Russian and East European Psychology* 48 no. 1 (2010), 18.
[39] Dewey and Bentley, 'Knowing and Known', 187, note 1.
[40] Dewey and Bentley, 'Knowing and Known', 133.
[41] Dewey and Bentley, 'Knowing and Known', 192.
[42] Marx and Engels, *Werke Band 3,* 6.

tion of the minimal unit of analysis when using the category of *pereživanie*, we cannot stop at the mathematics task in the consideration of what is included in the environment, or the classroom, or school. This is so because our observation are likely to differ dramatically if Sylvia left home saying, 'I can't wait for today's mathematics class' versus saying, 'Do I have to go to school? I don't feel like it'. That is, we have to consider the whole person, her inclinations, needs, and motives. I return to this issue in chapter 9.

Importantly, *pereživanie* and the emotional-volitional tone that is characteristic of it make an event absolutely unique. This is so even though, as seen in Fig. 6.3c, *perezivanie*$_1$ and *perezivanie*$_2$ of the two persons involved have the same extent. The two pereživanija are the same and different simultaneously. Whereas such a statement makes no sense in classical logic, it is typical of plurality of the one in the Spinozist-Marxian take, where we always find the 'concrete unity/identity [edinstvo] of mutually exclusive opposites'.[43] That is, whereas there is an objective event (environment) that can be given completely in natural scientific (physical) terms, *pereživanie* captures the individual's involvement and absolutely unique (particular) refraction of it (event). *Pereživanie* therefore inherently has a dramatic quality, where the individual is part of events to which it contributes and that it undergoes. The doubling that Marx and, following him, Vygotsky write about is to be understood only in an analytic sense. All we need to do is to recognize that 'our primary experience is dramatic, and that, if one considers things historically, it is not to nature that I superpose human significations [Fr. *significations,* 'meanings'], it is to the contrary nature that I reveal behind the latter'.[44] In keeping the dramatic nature of Life in mind, we inherently employ *pereživanie* in its transactional sense, the way in which Life actually is: even the most incomplete scenes of everyday life still imply human beings in their totality, that is, they imply the societal character of human essence. Traditional psychology, on the other hand, reduces experience to processes that no longer have the form or feel of our most everyday actions. One may see in ethnomethodology a singular attempt in reconstituting that renders to what we do its everyday feel.

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[43] Evald V. Il'enkov, *Dialectical Logic: Essays on its History and Theory* (Moscow: Progress Publishers, 1977), 320; Evald V. Il'enkov, *Dialektičeskaja logika: očerki istorii i theorii* (Moscow: Izdatels'stvo politi Izdatels'stvoeskoj literatury, 1984), 257.

[44] Georges Politzer, 'Les fondements de la psychologie: psychologie mythologique et psychologie scientifique', *La Revue de la Psychologie Concrète* 1 (1929), 27.

7

Affect and Emotion

Mathematics educators have shown interest in affect and emotion – as expressed, for example, by the 2006 special issue of *Educational Studies in Mathematics* 26 no. 2 on affect. Despite this apparent interest, the predominant number of studies still focus on intellectual issues at the expense of the inherently affective dimensions of life that from a cultural historical position must not be left out of theory unless it is to be fundamentally flawed. 'Quantitative' studies inherently miss the phenomenon because each variable is only an external manifestation of some phenomenon and, as philosophers from Baruch Spinoza to Georg Friedrich Hegel and Karl Marx showed, there cannot be but spurious relationships between such externalizations. Jean Piaget – though recognizing that it is impossible to find behavior solely composed of affective or cognitive 'elements' – nevertheless keeps them separate 'because, although cognitive and affective factors are indissociable in an individual's concrete behavior, they appear to be different in nature'.[1] He uses a comparison with an automobile according to which the structure of the intellect is to affect as the structure of the motor is to gasoline. Piaget concludes that the latter drives the former but does not change its structure. Spinoza would have told him that this conclusion is a truism, because the two things related are already external to and therefore inherently cannot affect each other. The Spinozist-Marxian Vygotsky would have taken a radically different perspective on what he denounced to be an age-old problem of psychology; but he never lived long enough to work out his position. Yet in much of the mathematics education literature we see an equal separation of affect (emotion) and intellect, as research commits precisely to what Vygotsky already denounced to be erroneous. This orientation is especially apparent in the affect-related research focusing on 'attitudes' and 'beliefs', which tend to be assessed using questionnaires (instruments) or interviews, that is, by way of intellectualizations.

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[1] Jean Piaget, *Intelligence and Affectivity: Their Relationship During Child Development* (Palo Alto: Annual Reviews, 1980), 3.

One of the fundamental flaws Vygotsky points out to be existing in the psycho-
logical literature of his time is the interpretive route, where *talk about* affect and
emotion – which constitutes an intellectualization – is equated with affect and
emotion that clearly have material origins that predate human (intellectual) con-
sciousness. Thus, when investigators use focus group interviews[2], then the data
constitute *talk about* emotion, which, as all talk, is characterized by socioculturally
specific narrative forms that always are produced *for* the other. In a strong sense,
therefore, such talk reflects the general properties of emotion-related language ra-
ther than the specific emotions related to the chosen participants, their age and
gender, or the specific mathematics course taken.[3]

Other research reduces affect and emotions to the individuals independent of
their function in the situations of interest. The results are then pure speculation.
Thus, for example, one study makes statements such as '**F's body seems to stiffen
at this point, suggesting resistance or fear/anxiety, possibly in response to T's
claim**' and 'his use of the first person may indicate withdrawal from the group,
positioning himself as a *solitary worker*. T's positioning here, and at several other
points, is ambiguous. **This may indicate T's feelings of isolation. However, he
appears cool with no overt indicators of emotion**'.[4] Readers will have no prob-
lems identifying the accumulation of speculative interpretations in these phrases:
'seems to stiffen', 'suggesting', 'possibly in response', and 'may indicate'. How
does one identify emotions if there are 'no overt indicators of emotion'? Although
there are studies that situate themselves in the conversation analytic literature, the
focus on the individual remains. For example, one studies describes a classroom
exchange in this way: 'At this point, Jaakko openly displays his emotions and dis-
appointment and exclaims loudly that he does not know. Even if the teacher has
been raising the stakes up to this point … she now responds immediately to his
emotional expression, and gives him the right answer with an overlapping turn'.[5]
Here, there are two agents who interact rather than a *social* phenomenon that plays
itself out in the public arena, and where the specific participants are but staff bring-
ing the phenomenon to life. That is, affect tends to be treated in terms of the feel-
ings of an individual person. But such an approach does not explain the inherently
social dimensions of affect. In fact, it has been shown that we cannot develop any

[2] Gustavo Martínez-Sierra and Maria del Socorro García-González, 'Undergraduate Mathematics Stu-
    dents Emotional Experiences in Linear Algebra Courses', *Educational Studies in Mathematics* 91
    (2016): 87–106.

[3] For example, a study in the context of environment and environmental protection shows that the talk
    of 14–15 year old students reflects the general discourse of their society on the topic. See Albert
    Zeyer, and Wolff-Michael Roth, 'A Mirror of Society: A Discourse Analytic Study of 14-15-year-old
    Swiss Students' Talk About Environment and Environmental Protection', *Cultural Studies of Science
    Education* 4 (2009): 961–998.

[4] Jeff Evans, Candia Moorgan, and Anna Tsatsaroni, 'Discursive Positioning and Emotion in School
    Mathematics Practices', *Educational Studies in Mathematics* 63 (2006), 217, original bold and ital-
    ics.

[5] Liisa Tainio, and Anu Laine, 'Emotion Work and Affective Stance in the Mathematics Classroom:
    The Case of IRE Sequences in Finnish Classroom Interaction', *Educational Studies in Mathematics*
    89 (2015), 83.

notion (conceptualization) of affect unless we look at ourselves through the eyes of the generalized other (the Other).[6]

The notions affect and emotion often are not clearly defined and are used synonymously even when investigators recognize that psychologists distinguish between the terms taking the former to cover a broader range of phenomena, including 'pre-conscious states of mind' and the latter to refer to 'culturally recognized states of the mind'.[7] It is apparent that both affect and emotion are treated as 'states of mind', which is contrary to the holistic take that we see in the Spinozist take that Vygotsky begins to articulate during the last months of his life. The introduction to the *Educational Studies in Mathematics* special issue suggests that affect covers beliefs, attitudes, emotions, and values.[8] These, then, are used as types of affect that lie on a beliefs-(values)-attitudes-(mood)-emotion continuum, characterized by traits from more durable, less intense, and more cognitive-reflective to more transitory, more intense, and more affectively charged.[9] This again is different from the way in which Vygotsky goes about it, where affect and intellect are qualitatively different manifestations of an inaccessible whole rather than being gradations of the same variable (affect). It is also different from the materialist phenomenological take, which recognizes three aspects in the constitution of the body: the constituting body, the constituted body, and the *non-constituted* original flesh that hides the invisible substance of life.[10] As Edmund Husserl showed in his *Cartesian Meditations*, the constituting body and the constituted body are intertwined. With respect to affects of the 'higher psychological sphere', such as 'the behavior of the irate, the cheerful', I cannot identify in the behavior of others unless I take the perspective of the other on my own behavior.[11] This interlacing of the two bodies 'marks the impossibility of a pure presentation, or also the presence of the *alter ego* at the very heart of the *ego*'.[12]

There are studies related to mathematics education that pursue such alternative lines; and they do so on the ground of cultural-historical studies. Although going versions of cultural-historical activity theory do not thematize affect in general and emotions specifically – perhaps because no place was made for them in the triangular representations that have become the quintessence of the theory in the Anglo-Saxon literature – *A Cultural-Historical Perspective on Teaching and Learning*

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[6] Didier Franck, *Chair et corps. Sur la phénoménologie de Husserl* (Paris: Éditions de Minuit, 1981).

[7] Tainio and Laine, 'Emotion Work', 68.

[8] Rosetta Zan, Laurinda Brown, Jeff Evans, and Markku S. Hannula, 'Affect in Mathematics Education: An Introduction', *Educational Studies in Mathematics* 63 (2006), 115–116.

[9] Martínez-Sierra, and García-González, 'Emotional Experiences', 91; and Jeff Evans, 'Affect in Mathematical Thinking and Learning; The Turn to the Social: Sociocultural Approaches Introduction: Recent Developments in Research on Affect', in *New Mathematics Education Research and Practice*, eds. Jürgen Maasz and Wolfgang Schloeglmann (Rotterdam: Sense Publishers, 2006), 234.

[10] Michel Henry, *Incarnation: Une Philosophie de la Chair* (Paris: Éditions du Seuil, 2000), 221–222.

[11] Edmund Husserl, *Gesammelte Werke Band 1: Cartesianische Meditationen und Pariser Vorträge* (The Hague: Martinus Nijhoff, 1973), 147–149.

[12] Franck, *Chair et corps*, 157.

builds on the work of the Vygotsky student and collaborator Alexei N. Leont'ev.[13] In that take, the 'objectivity of activity is responsible not only for the objective character of the images but also the objectivity of the needs, emotions, and feelings'.[14] Accordingly, emotions and feelings are a reflection of the objective, societal, generalized need-related activity, on the one hand, and the needs and goals of the individual subject, on the other hand; that is, they reflect the relation between activity-related, society-determined motives and the subject's anticipated success in realizing it. The problem in mathematics education is that in contrast to the kinds of work-related situations where activity theory tends to be used, students in mathematics classes *cannot know* the object/motive of the activity-realizing task until after the lesson has achieved what it is intended to achieve: student understanding of some mathematical issue. Explicitly oriented against openly hedonistic theories of human life, cultural-historical activity theory conceives of the integral role of emotions in human behavior, where they are 'not subordinated to activity but appear to be its result and the "mechanism" of its movement'.[15]

In cultural-historical activity theory, conscious motives drive activity at a societal level – car manufacture has as object/motive the production of cars from raw materials, which may themselves be produced by means of outsourcing. Schooling has the object/motive the production of grades and grade reports.[16] If affect were only related to the relationship of motive and the current state of the activity of the subject, who, in participating, has taken up the motive, then affect would only be a byproduct of the intellect. But why would intellect charge itself with troublesome affect? Why would evolution have selected affect if it were only a product of the intellect?

There are other aspects to the phenomenon, as readers certainly will have experienced. Thus, we may feel down, or sad, or self-deprecating when coming to work, a sports event, or a party only to find ourselves all of a sudden with very positive emotions; we may not feel like doing our regular sport but, if we get ourselves to start, we will find ourselves, after about 30 minutes, in an emotional high. Here, the participation in activity, even though tinged negatively, co-produces positive levels of affect. Thus, even though a student might be frustrated in the course of working on a task because he does not seem to make progress, or because he does not understand what the task is asking from him, continuing is the only hope to overcome the current state of negative affect. This hope itself is a form of emotion, though a precarious one, to overcome the negative emotions to end up with the satisfaction that comes with succeeding in a task.

Another study in the cultural-historical tradition explicitly investigates mathematics anxiety as a particular form of affect in the context of mathematics educa-

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[13] Wolff-Michael Roth, and Luis Radford, *A Cultural-Historical Perspective on Mathematics Teaching and Learning* (Rotterdam: Sense Publishers, 2011).

[14] Alexei N. Leont'ev, *Activity, Consciousness, and Personality* (Englewood Cliffs: Prentice-Hall, 1978), 54.

[15] Leont'ev, *Activity*, 120.

[16] See, for example, Wolff-Michael Roth and Michelle K. McGinn, '>unDELETE Science Education: /lives/work/voices', *Journal of Research in Science Teaching* 35 (1998), 400–401.

tion.[17] Grounded in the works of Vygotsky, the study situates affect by understanding it as a manifestation of *pereživanie* (see chapter 6). Here, affect no longer is a characteristic of the person but a characteristic of the relation between the person and her environment. As a result, affect no longer can be explained by characteristics of the person, the environment, or the *inter*action between person and environment. Instead, affect is a reflection of the transactional {person | environment} unity/identity, which also includes intellect and practical action.

In this chapter I articulate a Spinozist-Marxian approach to affect. Affect and intellect are different manifestations of a whole – i.e. the production of material life – and whatever we can be consciously aware thereof inherently is social. Moving in this way allows us to think about how relations with others and the classroom culture as a whole have affective qualities with important functions in the affective relation of the individual to mathematics. Vygotsky notes – in the context of his newly developed Spinozist stance on emotions – that affect is capable of vanquishing affect. It does so not by means of the intellectualist route – talk about affect is different from affect-in-situation – but by way of its inherently cultural and bodily nature.

Developing the Cultural-Historical Approach

An important aspect of Vygotsky's work – which continuously evolved in the course of his life – was the Spinozist take that emerged near his death. Even though it figured in the introductory chapter of *Thinking and Speech* devoted to method-related issues as a way of overcoming the psychophysical (mind–body) problem that has been and continues to plague psychology, this part of the cultural-historical agenda was not really taken up in recent work. In another text from the same period, 'The Teachings of Emotion', Vygotsky wrote an opening for his work to come that was to take a Spinozist-Marxian approach integrating intellect (consciousness) and affect. Whereas the text is devoted to the analysis of the Cartesian split in psychology, which is embodied in the physiological or interpretive approaches to human psychology, we get an inkling of where he sees himself to be heading in the opening chapter of *Thinking and Speech*. Here he summarizes his readings stating that traditional psychology had separated intellect from affect. He proposes instead to use unit analysis suited to deal with the 'dynamic meaningful system that constitutes a *unity of affective and intellectual processes*', which allows uncovering that 'every idea contains some remnant of the individual's affective relationship to that aspect of reality which it represents' and 'see[ing] the relationship between the individual's needs or inclinations and his thinking'.[18] In his

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[17] Wolff-Michael Roth and Margaret Walshaw, 'Rethinking Affect in Education from a Societal-Historical Perspective: The Case of Mathematics Anxiety', *Mind, Culture and Activity* 22 (2015): 217–232.

[18] Lev S. Vygotsky, 'Thinking and Speech' in *The Collected Works of L. S. Vygotsky: Volume 1: Problems of General Psychology* (New York: Plenum Press, 1987), 50.

approach, thinking and emoting are two different manifestations of the same life of the individual, including his/her motives, interests, and inclinations. As apparent in 'The Teaching of Emotions', he intended to develop an approach grounded in his Marxian re-reading of Spinoza.

When Vygotsky does his historical psychological studies on the teaching of emotions, he articulates himself as not being in the business of reifying the truth and power of Spinoza's *Ethics*. The *Ethics* does not provide a ready theory of emotion for the present day. Vygotsky instead pursues an agenda of testing the ideas the *Ethics* contains in the light of contemporary scientific evidence. This process affords separating those aspects that withstand the test of time and experiment and those ideas that are false. In true Spinozist manner, *Thought* – understood as the collectively produced best of scientific knowledge of the day – thereby develops. Most importantly, perhaps, is the deeply rooted antithesis that the *Ethics* constitutes with respect to the Cartesian system. The work on emotion embodies 'one of the greatest revolutions of the spirit and a catastrophic overturn of the [Cartesian] system of thought'.[19]

The general definition of the emotions only appears at the end of the third part of the *Ethics*, in which Spinoza deliberates on the origin and nature of the emotions. His general definition is quite complex, and, unsurprisingly, is followed by an explication. 'Emotion, which is called a passivity of the soul, is a confused idea, whereby the mind affirms concerning its body, or any part thereof, a force for existence (*existendi vis*) greater or less than before, and by the presence of which the mind is determined to think of one thing rather than another'.[20] This definition reaffirms the integral and irreducible relation holding together mind and body within one thinking body. In contrast to other materialist approaches, the Spinozist-Marxian take does not equate the thinking body with the material body. Instead, extension (body) and thought are two manifestations of a unitary phenomenon. Emotions are manifestations of the greater or lesser power of the thinking body to *act*. There is an idea, which constitutes the actual form or reality of the affect; and this idea manifests or expresses the state or disposition of the body. Spinoza does not suggest that the mind actually compares two states of the disposition of the body with respect to its power to act; and he insists upon the fact that the emotion manifests or affirms something of the body.

Spinoza theorizes emotions as integral to human activity and the thinking that comes with it. Emotions simultaneously pervade the activity of the human body and the correlative thinking. Thus, '*whatsoever increases or diminishes, assists or checks, the power of activity of our body, the idea of the said thing increases or*

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[19] Lev S. Vygotsky, 'The Teaching about Emotions: Historical Psychological Studies', in *The Collected Works of L. S. Vygotsky. Volume 6: Scientific Legacy* (New York: Kluwer Academic Publishers, 1999), 74.

[20] For the Latin version see Benedicti de Spinoza, *Ethica: Ordine Geometrico Demonstrata et in Quinque Partes Distincta in Quibus Agitur*, chapter 3, available at, http://users.telenet.be/rwmeijer/spinoza/works.htm.

diminishes, assists or checks the power of thought of our mind'.[21] Pleasure, pain, and desire constitute the fundamental emotions. In considering desire, Spinoza notes that it is not different from appetite but for the fact that in desire, humans also are conscious of their appetite – which leads him to define desire as 'appetite accompanied by the consciousness thereof'.[22] We observe here what later appears in Marx and Vygotsky as the doubling of experience, where a primary phenomenon is accompanied by consciousness. This also gives us an inroad to understanding why interviews about emotions in the mathematics classroom do not get at the emotions that are at the core of mathematical activity. It is in any case only an after-the-fact investigation into what stands out through the veil of what has happened since the mathematical activity of concern.

Spinoza distinguishes between 'titillation' and 'cheerfulness', both emotions of pleasures involving mind and body, but the former only affects parts of a person, whereas the latter affects the whole person. A parallel situation exists for two emotions of pain involving body and mind, 'anguish' and 'melancholy', the former only affecting parts of the person, the latter the whole person.

For Vygotsky, Spinoza developed a take on the emotions that 'gave a guiding beginning, fruitful not only for the present, but also for the future of our science'.[23] In his view, the empirical evidence in favor of a dynamogenic effect of emotions lends support to the Spinozist conception of 'affect as states of the body which increase or decrease the capacity of the body itself for action, promote or limit it, as well as ideas of these states'.[24] Pursuing an agenda oriented toward overcoming the intellectualist take on affect, Vygotsky notes that 'language is not made for emotions. That is the difficulty of expressing and studying emotions <u>is that as it should be</u>? ?? And giving the n[a]me to an emotion changes the emotion (*prise de conscience*) not only in the sense of a "lie" (Tiutchev) but also of the flow: i.e., behind this difficulty lies the real phenomenon of the change emotion – word!'[25]

The Spinozist-Marxian take is holistic because the activity of the body, thinking, and emotions all are manifestations of the same whole – which, in Vygotsky's work, leads to the postulation of *pereživanie* as the all-integrating category. The holistic approach manifests itself, for example, in the definition of the 'passive emotions' as those that involve the relation of the person to an external object. A passive emotion, therefore, cannot be understood in terms of the nature of a person but by necessity includes the nature of the object. Thus, 'the force and increase of any passive emotion and its persistence in existing is defined … by the power of external causes compared with our own power', 'not by the power whereby we

[21] Baruch Spinoza, 'Ethics', in *Complete Works,* trans. Samuel Shirley (Indianapolis: Hackett Publishing, 2002), 284.

[22] Spinoza, 'Ethics', 284.

[23] Vygotsky, 'The Teaching about Emotions', 121.

[24] Vygotsky, 'The Teaching about Emotions', 78. See also Spinoza, 'Ethics', 278.

[25] Lev S. Vygotsky, quoted in Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive: New Findings – Notebooks, Notes, and Scientific Journals of L.S. Vygotsky (1912–1934)', *Journal of Russian and East European Psychology* 48(1), 51.

ourselves endeavor to persist in existing'[26] – the very opposite of what constructivists tend to suggest. The situation is much like that with the transactional definition of aptitude, where the understanding of the person characteristics in performance requires specification of the environmental characteristics, and understanding the effects of environmental characteristics on performance requires specification of the person characteristics.[27] This is a typically Spinozist take, according to which 'the explication of the nature of every passive emotion must necessarily include an expression of the nature of the object by which we are affected'.[28] Thus, if two objects A and B both give rise to pleasure, then the two emotions of pleasure differ, for one includes characteristics of the object A and the other characteristics of the object B. Spinoza makes the parallel case for persons, according to which the emotion of a person A and that of a person B in relation to some object require the specification of personal characteristics, which will differ for A and B. In this way, the 'passive emotions' – pleasure and pain as the passive transitions of the mind to greater and lesser perfection, respectively – Spinoza provides a transactional definition. Much as the community has turned to 'situated cognition' and 'cognition in practice', we may therefore turn, following the Spinozist Vygotsky, to 'situated emotion' and 'emotion in practice'.

Besides the passive emotions, Spinoza recognizes those related to the state of being active. Thus, the basic emotions of pleasure and pain arise in the context of the mind, such as when pleasure arises from conceiving an adequate idea. Desire is specific to humans – therefore, essentially human – and the human essence because it involves both an orientation toward the object of activity and the consciousness thereof.

Some of the emotions that we can easily relate to mathematics education are inclination, aversion, hope, fear, confidence, and despair. According to Spinoza, inclination is the pleasure, aversion the pain, accompanied by the idea of a thing that is indirectly the cause of the pleasure or pain. Similarly, hope and fear are the inconstant pleasure and pain arising from the idea of a thing future or past, of whose outcome we are in doubt. Finally, confidence and despair are the pleasure and pain arising from the idea of a thing future or past, concerning which reason for doubt has been removed. Hope and fear therefore are interrelated, for there is no hope without fear, and no fear without hope. For as long as we hope, we also fear that we may not succeed, and as long as we fear that we may fail, there is also hope that we may succeed. Confidence and despair arise from hope and fear, respectively, when the associated uncertainties have been decreased or entirely removed. In a parallel way, joy and disappointment arise when the outcomes of activity are contrary to our fears and hopes, respectively.

There are emotions arising from principally intellectual endeavors. Thus, the phenomenon that has something singular stand out, affecting principally the mind

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[26] Spinoza, 'Ethics', 325.

[27] Corno et al. specify in this manner the transactional take of Richard E. Snow. See Lyn Corno et al., *Remaking the Concept of Aptitude: Extending the Legacy of Richard E. Snow* (Mahwah: Lawrence Erlbaum Associates, 2002), 216.

[28] Spinoza, 'Ethics', 307.

or thought about the phenomenon that Spinoza refers to as *admiratio*, Latin for 'wonder', 'admiration'. It arises when something is strange or surprising. This aspect may be cultivated in those mathematics classrooms that are able to invite students in explorations without the threat of negative sanctions weighing on them in the case that they do not 'construct' on their own or with peers what the curriculum states as the expected outcomes. On the negative side, the moderns' anxiety, such as in mathematics anxiety, can be understood on the basis of Spinoza's pain that the mind feels when confronted with its own impotence.

'Emotions are not subordinated to activity but appear to be its result and the "mechanism" of its movement'.[29] Productive, societal activity, to reiterate, is the sense-constitutive minimal whole that has all the characteristics of society and, therefore, of consciousness and personality. Emotions reflect activity, as 'the emotional-volitional tone relates precisely to the *whole* concrete and once-occurrent unity in its entirety'.[30] Furthermore, they are not immediate psychological reflections of objective activity as such. Instead, emotions reflect the relationships between the object/motives of societal activity in which the individual is engaged and (the possibility of) success of realizing the activity-constituting actions. This function was already formulated in the *Ethics*, which states that '*besides the pleasure and desire that are passive emotions, there are other emotions of pleasure and desire that are related to us insofar as we are active*'.[31] This form of reflection is not of the intellectual kind, not of consciousness that appears to us in verbal form, but an affective-sensory reflection, as in *pereživanie*. Marx is concerned, in the theses on Feuerbach, with labor as a real, sensory involvement of humans in their world. The gnostic and affective moments of activity are united in this primary sensuality. Emotions arise in objective conditions, marking in emotion-specific ways these situations and individual objects.

A Case Study of Affect in Mathematics

In the following, I draw on a fragment from a fourth-grade mathematics curriculum but not only investigate its relation to affective issues in three students simultaneously but also do so from a late Vygotskian Spinozist-Marxian position articulated above that heretofore has not been available in the mathematics education literature.

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[29] Leont'ev, *Consciousness*, 120.
[30] Mikhail M. Bakhtin, *Toward a Philosophy of the Act* (Austin: University of Texas Press, 1993), 36.
[31] Spinoza, 'Ethics', 309.

Ethnography of a Lesson Fragment

The case materials in this section derive from a fourth-grade mathematics course that participated in an experimental curriculum designed to solicit the early development of algebraic notions and understandings. Though experimental, the curriculum was designed such as to meet the provincial curriculum guidelines, which, among other topics, specify student learning for that grade in this way:

– Patterning and Algebra: generating patterns that involve addition, subtraction, multiplication, and reflections;
– Representing: create a variety of representations of mathematical ideas (e.g., by using physical models, pictures, numbers, variables, diagrams, graphs, onscreen dynamic representations), make connections among them, and apply them to solve problems;
– Patterns and Relationships: extend, describe, and create repeating, growing, and shrinking number patterns (e.g., 'I created the pattern 1, 3, 4, 6, 7, 9, … I started at 1, then added 2, then added 1, then added 2, then added 1, and I kept repeating this'.);
– Patterns and Relationships: connect each term in a growing or shrinking pattern with its term number (e.g., in the sequence 1, 4, 7, 10, … the first term is 1, the second term is 4, the third term is 7, and so on), and record the patterns in a table of values that shows the term number and the term.[32]

The particular task that the students in this French immersion class[33] were doing involves the story of a girl, who receives a piggy bank with $6 base amount and in which she deposits $3 each week (Fig. 7.1). Readers can easily see that the task involves the processes described in the curriculum, including (a) representing the piggybank and the dollar amounts in the story by means of a goblet and chips using a physical model; (b) generating and describing patterns that involve addition (second row in table of values) and multiplication (third row in table of values); (c) using the term number (first row) to generate a pattern in the third row; and (d) generating patterns.

The camera follows the events around one group of four desks where there are three students, Aurélie, Thérèse, and Mario (Fig. 7.2). At one point, the teacher, following a raised hand on the part of Mario, comes to the group of desks and ends up in an exchange with Mario, orienting herself only occasionally to the two girls. The goblets are standing in front of Mario, who has been counting out and placed the chips.

Thérèse apparently seems to be in the know, a fact displayed *for* and taken up *by* the others. She laughs while the others are struggling, speaks with an intonation

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[32] Ontario Ministry of Education, *The Ontario Curriculum Grades 1–8: Mathematics* (Toronto: Author, 2005), 64, 65, 73; available at http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf.

[33] Students in such classes, if they are not immigrants, tend to be native English speakers who take all of their classes but reading (English) in French, thereby are (supposed to be) learning this second language by immersion in it.

Problem

For her birthday, Marianne receives a piggybank containing \$6. She decides to save \$3 per week. At the end of the first week she says to herself, "I have \$9!"

Questions:

   a. Model the problem by means of boxes and chips until the sixth week

   b. Fill up the table of values below

Number of week	1	2	3	4	5	6
Amount saved (\$)	6	+6				
Or	6	2x +6	3x +6	x +6	x +6	x

Fig. 7.1 The English translation of the task assigned to the students on this day

that manifests confidence. In reply to a query about what to do next, she orients her peers to the still empty table of values on the worksheet. When both Aurélie and Mario ask her what she is doing, the reply begins with a confidently intonated interjection, which is followed, after a pause with an invitation to look at the board. When the teacher eventually arrives and asks whether they 'are in the course of discussing within their group', Mario replies that Thérèse 'just has taken off' and invites the teacher to 'look at how they, the two girls, have already written stuff'. In the following fragment, already while Mario and Aurélie are articulating what will have been treated as queries, Thérèse has been pointing toward the board.

Fragment 7.1

01 M: yes, what are you doing
           ((*T starts pointing toward the chalkboard*))
02 A: yea (0.3) what are you doing thérèse, where are you at. okay we dont have the same thing as you
03 M: yea::
04 T: ((*confidently*)) uh hn (1.1) look at the board.
05   (0.8)
06 A: dude >we dont have the same thing as you<

   In contrast to her two peers, whose statements are treated in her own turn as queries, Thérèse does appear to know what to do, at a minimum, what she is doing. Turn 06 treats the statement in this way, as it does not deny the appropriateness of the preceding turn but instead articulates a difference with the task of the others ('we'). A few turns following the fragment, Mario also produces a statement with respect to an action on the part of the teacher, the results of which are still seen on the board (where he is looking at the time): 'She just added to the side'.

   In the course of the events, Thérèse rapidly fills up the table. She moves about in ways that may be glossed by means of the adjective 'confident'. Indeed, others in the situation treat her behavior in this manner, as both Aurélie and Mario turn to

Fig. 7.2 Dry brush rendering of an event where three fourth-grade children during a task that asks them to use goblets and differently colored chips to model the successive states of a piggy-bank that starts with a $6 base amount and in which $3 are deposited each week. From left to right are Aurélie, Thérèse, and Mario (pseudonyms)

Fig. 7.3 After repeatedly manifesting frustration and querying others what they are doing, Aurélie has placed her head on the desk. Thérèse calls on her with an intonation expressing something like, 'Aurélie, come on', and then writes into her peer's table of value

her and her worksheet. In one situation, Aurélie points to the yellow highlighted numbers and Mario asks what to do now. Thérèse replies, 'I know, but you can't see it just like this, though you can see it'. Her torso is upright, and the right arm and hand configured in the manner people in the know often do when expressing regret that someone else is incorrect. When others query, Thérèse says 'chuggy chuggy, just copy me'. That is, even if she does not have in her conscious mind that she is moving ('chugging') along, the very expression manifests that continuous movement that the word stands for. She may indeed become *consciously* aware of her moving along only in and with saying 'chuggy chuggy'. She can see what has to come next, though it may not be immediately transparent what was to be filled into the cell of the table of values. Similarly, when Aurélie has already manifested frustration repeatedly – pounding on the desk, laying the head to rest on the desk (Fig. 7.3), and, in a wining intonation, made statements concerning not understanding – Thérèse, in a drawn out intonation that may be heard as in the gloss, 'Aurélie, Aurélie, come on'. She then turns her own worksheet toward Aurélie and then writes into the first cell of the second row of her friend's worksheet (Fig. 7.3).

In the relation with the teacher, as shown in Fragment 7.2, Thérèse (T) also is produced as being in the know. Thus, in a sequence where the topic is the content of the piggybank during the first week, after several apparently muddled replies, the teacher (t) states the composition, which the next turn affirms to have been said before (turn 065). This is treated as a claim by the statement that asks for reason, which the next turn provides qua reply. By taking up the content of the preceding statement in a constative manner (rather than with a rise in intonation so that it could be heard as a question), it de facto evaluates it as correct. This evaluation further manifests itself in the fact that the querying continues rather than remaining on the same issue.

Fragment 7.2
064 t: =*it*s three plus six:.
065 T: ive said °°[that is what ive said]°°
066 t: [why three plus six:.]
067   (0.34)
068 T: because it equals to the deposit of the first week she has nine.
069 t: it *eq*uals to nine the first week.

Here, then, the positive affective manifestations produced in and through Thérèse's behavior are not simply the results of a researcher interpretation but their production is taken up in the behavior of others and, therefore, in the unfolding of the lesson as realized in this group. Not only here but in general, affect is objectively written all over situations, which is why different observers of some situation may agree that it 'was somber', that someone 'was embarrassed', or, in the present context, that Aurélie was frustrated, even dejected, disheartened.

Throughout this event, Aurélie's behavior stands in a stark contrast to that of Thérèse. During the entire recorded lesson fragment, Aurélie's voice sounds plaintive, for example, after Mario suggests that in the first cell they have to write '3 + 6', 'où tu voies ça? [Where do you see this?]' (Fig. 7.4). There are considerable pitch variations, which, in the example, range from 244 to 625 Hz. Thus, in the word 'ça [this]' alone, the pitch begins at 437 Hz, rises to 604 Hz, and then descends to 244 Hz all the while much of the time the speech intensity is nearly constant; there is therefore a pronounced contour. That one syllable also extends over 0.6 second. Together, these variations give the voice its 'wining' quality. As an ensemble, the parameters are consistent with the reported speech correlates of displeasure and disgust.[34] When Thérèse invites, 'just copy me,' the reply declines stating incapability 'I can't [j'peux pas]', with the pitch going from 270 to 822 to 283 Hz in a matter of 0.4 seconds. It is heard as a very sharp and strident 'can [peux]' characteristically associated with fear/terror but not with any other form of emotion.[35]

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[34] Klaus R. Scherer, 'Vocal Correlates of Emotion', in *Handbook of Psychophysiology: Emotion and Social Behavior* (London: Wiley, 1989), 193.
[35] Scherer, 'Vocal Correlates', 193.
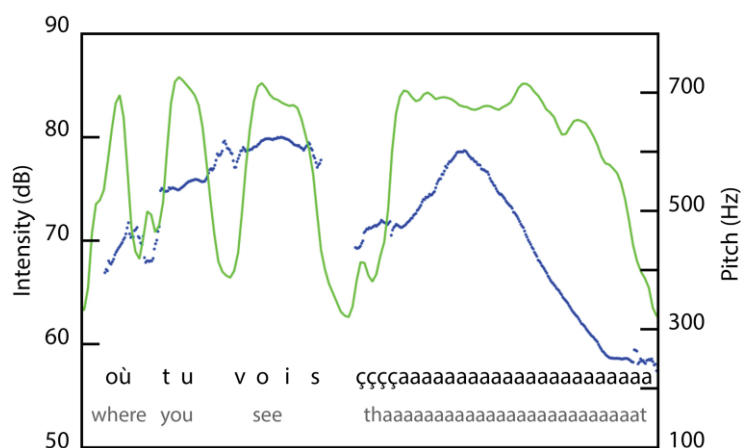
Fig. 7.4 The speech parameters – including speech intensity, pitch, and pitch contour – are consistent with the correlates of dejection and frustration, and, frequently, can be heard as 'wining'

Throughout the lesson, Aurélie frequently sits in ways that manifest disengagement (Fig. 7.5). She throws herself backwards against the backrest of the chair, then 'slouches'; she also places the head on the desk as if sleeping, or props the head with the right hand without simultaneously attending to the task. The teacher, after having arrived at the group of desks admonishes Aurélie to sit properly, literally saying, 'as required [comme il faut]'. Aurélie repeatedly pounds on the table, and then either puts her head on the desk as if going to sleep or sleeping, or leans back into a reclined position against the backrest. But not all of the behavior expresses tuning out and turning off. Instead, after frequent expressions of frustration, and after having already copied something from Thérèse's worksheet, and while watching the teacher in an *obučenie* [teaching | learning] sequence with Mario, we observe evidence of counting with her fingers, with pauses following every set of three (Fig. 7.6).

Dejection and frustration also are apparent in the speech contents, which constitute articulation of consciousness for others as much as for Aurélie herself. Initially, she still makes a few comments (e.g. 'well, we only have to go to twenty-four') and asks questions about what is going on: 'why did you just add [the red chips] to the blue ones?', 'Now what?', 'where do you see this', 'What are you doing Thérèse?' 'Where are you at? We don't have the same thing as you', 'I have no clue what she is doing', and 'This makes no sense!' Most importantly, perhaps, she states, 'I can't do it' and 'I do not understand, and I will never understand'. When she apparently does not understand, she invites others, 'we just continue to the next [item]'.

Whereas the teacher does not treat Aurélie's behavior as manifestation of a need for help, Aurélie's frustration clearly is apparent, and is taken up by Thérèse, who turns to her friend with apparent efforts to help, filling in the table or beginning an explanation after Aurélie makes a particular self-deprecating statement about her

Fig. 7.5 Dry brush rendering of different body positions and orientations on the part of Aurélie, which others treat as inappropriate (teacher) or as asking for assistance (Thérèse)
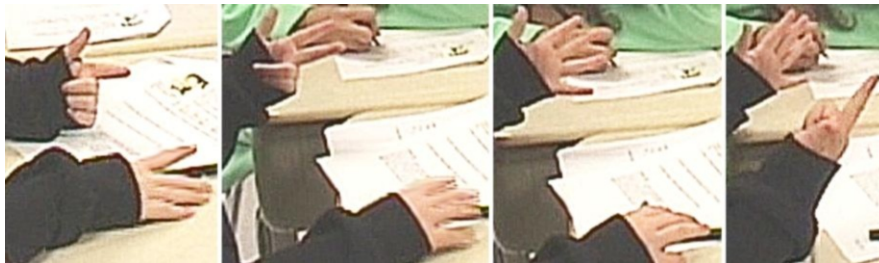
Fig. 7.6 Apparent finger counting, which the video reveals to be in groups of three, as there is a pause each time three fingers have been moved into place

present and future understanding (turn 29): We observe that there are pauses for Aurélie to pick up and reply (turns 31a, c), but, as she does not enter the floor, this also gives space for Thérèse to continue. The take is an affirmative 'yeah', though the rising intonation typical for a question also makes it possible to hear the statement as 'yeah, really?' (turn 33).

Fragment 7.3
029 A: ((*plaintive*)) i dont underst*and*; and I will n*eve*r understand.
030   (0.84)
031 T: here (0.30) you have to do
   a   (1.41)
   b   three plus six (0.60) yup.
   c   (1.79)
   d   <y=write whatever>
032   (1.29)
033 A: ye*ah*?

Turning our gaze now to Mario, the video provides evidence that he engages with the task and seeks to complete it. He does a major part of the modeling part of

the task (Question a, Fig. 7.1), counting out the chips and placing them into the goblets. For example, he 'thinks aloud' for others as for himself: 'Thérèse, what are you doing? You did it wrong. First week, she has but nine. We write, "nine". The second week, she has how much? And we write it. Third week, how much, how much …' But eventually, he begins to turn his torso around to be able to gaze in the direction of the teacher, who is currently working with another group off-camera. For over two minutes, he looks in the direction of the teacher, raises his hand, and momentarily drops it to attend to his worksheet only to raise the hand again. He does so until the teacher finally arrives. The invitation, 'so what is the question' comes to be paired with the statement, 'this is dumb. I don't understand'. This statement does not treat the invitation as a query about a question; instead, it treats it as an invitation to describe an issue, 'this [task] is dumb', and to state a problem, 'I don't understand'. The unfolding events will show that the teacher orients toward addressing the self-description.

Over the next five minutes, the video features an exchange that almost exclusively involves the teacher and Mario. The form of the relation is discussed in chapter 8 as *obučenie*, {teaching | learning}. In non-technical language that focuses on the teacher alone, we might gloss that part by stating that 'she helps him read the word problem and how to go about solving the translation from the goblet contents to the contents of the first and second row of the table'. In Mario's case, too, affect is written all over the unfolding event. We hear the sudden and dramatic rise of intonation in the word 'twelve [douce]' as impatience, especially as the content of the reply is self-evidently displayed before them in the contents of the second goblet. Despite their initial joint efforts, Mario manifests increasing frustration; and this frustration, the lack of understanding, and the failure to overcome it by means of the instruction is reflected in an apparent frustration on the part of the teacher. Frustration and dejection are sensible in the content of speech, voice, and positing and movements of body, as shown in Fragment 7.4. The two are currently concerned with the contents of the second goblet and table cell. The teacher states what *we* (but, as seen, not Mario) can hear as an invitation to articulate the composition of the $12 in the goblet (piggybank) (turn 166).

Fragment 7.4
164 M: wh*at* w*ell* l*ook*
165   (0.3)

166 T: twelve d*oll*ars cont*ain*s the::?
  a   (1.48)
  b   six d*oll*ars that we start w*ith*? (0.46) and how
         m*uch* money in the other two weeks be*for*e

167   (2.01)
168 M: what?
  a   (1.56)
  b   that makes- (0.80) i dont
         underst*an*d (.) th*ou*gh
169 T: °you dont understand
         that° its what i=m trying
         to help you understand

There is a long pause, which then is treated as failure to reply in offering to state the composition in terms of the starting amount of six dollars and the savings of three dollars in each of week 1 and week 2 (turn 166b). Despite the elaboration, there is another long pause, brought to a close by the interrogative 'what?', itself followed by an instructionally long pause. In this instance, as the teacher does not take the speaking floor, there is space for Mario to articulate an evaluation of the situation. The first two words appear to signal the coming of something like 'that makes no sense'; but he stops before completing a phrase, which opens up into a pause that is brought to a close with the statement 'I don't understand'.

The visual representation in turn 168b shows a two-handed gesture, as if Mario were holding something above worksheet. Already before, following an invitation to comment on the origin of the $12 amount, there is a similar gesture, both hands oriented toward the worksheet, palms up, the statement 'what, well, look' with an intonation of dramatic contour changes, a pitch much higher than normal (Fig. 7.7). Readers clearly see that the 'what' comes with an intonation typical of an outcry, 'What?' rather than as it might be produced as part of the phrase, 'What time is it?' The intonation within the sound 'regardes [look]!' twice rises sharply at higher than normal pitch, which is thus heard as an exclamation; and it falls together with the hands oriented toward the upper part of the worksheet where the story and table of values can be found. Later, when the hand movement toward the worksheet is repeated, his gaze is then moving from the sheet to her face. The reply, following a paraphrase of what has been said, then states the intent to help him understand (turn 169).

Eventually the event comes to a point when the teacher with excitement in the voice positively acknowledges the production of replies, and eventually develops to the statement, 'I think you understand now', which is affirmed by means of a head nod. Over the next 50 seconds, Mario works on his own, at one point bending over to look at Thérèse's worksheet, and then with confidence in his voice, says, 'Me, I understand now'.

At that point all three have completed their worksheets. Thérèse has done hers all on her own; Aurélie has copied from Thérèse, and Thérèse has written the contents of the first cell; and Mario participated with the teacher in their jointly produced work (Fig. 7.8). Frustration, as dejection, was manifest in verbal consciousness, body movements, and prosody. These results are not interpretations imposed
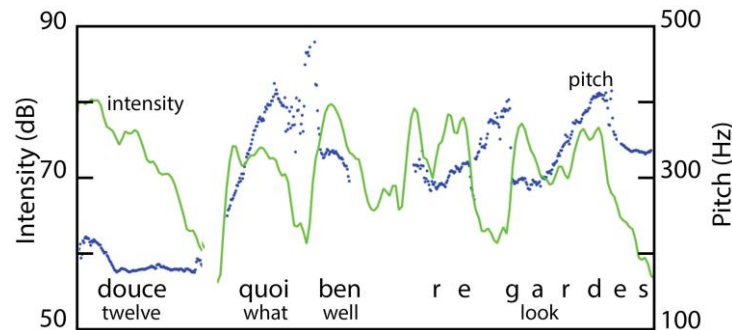
Fig. 7.7 Normal pitch level ('douce') compared to the pitch at a time during the task when Mario also verbally states not to understand. The two syllables of 'regardes' are much longer than the two preceding syllables

on the events but, as in the case of the two girls, other participants themselves acted upon the ways in which intellect and affect manifested itself in public – and, therefore, also, for the performers as well.

A Holistic Analysis

In his final, particularly Spinozist period of work, Vygotsky emphasized the need to conduct *unit analysis*, which he distinguishes from the analysis in terms of elements, pointing out that 'a psychology that decomposes verbal thinking into its elements in an attempt to explain its characteristics will search in vain for the unity that is characteristic of the whole'.[36] One such unit, as shown in chapter 6, is *pereživanie*. But when using this unit, we must not forget that in the forgoing description, the whole is not the task or lesson or mathematics. The whole may in fact be the entire life of any single participant. Thus, understanding the behavior of any single person, Mario, Aurélie, Thérèse, or the teacher, we may have to take into account each person's life as a whole. Thus, it may (but does not have to) be relevant that teaching is a priority in the life of the teacher; and what she does may differ if teaching were ranking only fourth, after family and other forms of activity. It may also be relevant that she has been participating in regular meetings – also involving the researchers and the research assistants – concerning the content of the experimental curriculum. In a strong sense, therefore, we need to know the personal histories of participants and situation to make sense of the historical antecedents of this event. This would allow us to understand why, for example, a student from a middle- and upper-class family exhibits behaviors in mathematics class that a working-class student, with very different trajectory of pereživanija, does not manifest. Thus, we notice that the whole we choose has to be larger than this task

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[36] Vygotsky, 'Thinking and Speech', 45.

Aurélie

1	2	3	4	5	6
3+6	3+3+6	3+3+3+6	3+3+3+6	3+3+3+3+6	3+3+3+3+6
3+6	2x3+6	3x3+6	4x3+6	5x3+6	6x3+6

Mario

1	2	3	4	5	6
3+6	3+3+6	3+3+3+6	3+3+3+3+6	3+3+3+3+3+6	3+3+3+3+3+3+6
3+6	2x3+6	3x3+6	4x3+6	5x3+6	6x3+a2

Thérèse

1	2	3	4	5	6
3+6	3+3+6	3+3+3+6	3+3+3+3+6	3+3+3+3+3+6	3+3+3+3+3+3+6
3+6	2x3+6	3x3+6	4x3+6	5x3+6	6x3+6

Fig. 7.8 Final contents of the table of values for each of the three participants in the lesson fragment

or lesson particularly in the case of Aurélie when she states, 'I don't understand and I will never understand'. In this articulation, we have – following the likes of Marx, Vygotsky, and Vološinov – a verbal statement that *is* consciousness for others and herself. It concerns the task, or, more precisely, the relationship between the materials at hand, as available on the worksheet with the story problem, and the anticipated outcome. But the statement goes further in that it projects into the future and beyond this task and onto the relationship between the demands of the mathematics curriculum and her own success in taking up and realizing the object/motive.

In a strong sense, if affect were to be intellectual alone, it would not be necessary at all. Realizing that she does not understand would suffice Aurélie as an evaluation of the situation. Why would thought require affect when it already realized that it is incapable of dealing with the task? What would be the function of affect if it were merely intellect translated into a different form? Why would evolution have selected affect as a function in the life of our species if it were only expressing what intellect already knows? These questions make it quite apparent that emotions need to be theorized because they manifest that there is always something at stake beyond intellect, and the possibility of failure thus is experienced as threat and is associated with the emotion of fear. It may therefore not come as a surprise to see Aurélie repeatedly manifest disengagement – in leaning back, closing the eyes, or placing the head on the desk literally moving away from doing something that might lead toward the completion or failure of the task. If I do not engage in the task, then I do not have to fear failing. But, in this case, I also do not at all accomplish the task. I avoid the task; and, to avoid punitive measures, I may do something else that nevertheless produces something that may be taken to be the results of my activity – such as copying the results that someone else has arrived at (Aurélie copying from Thérèse) or letting someone else do what has been

assigned to me (Thérèse writing into the first cell of Aurélie's worksheet). Readers will not have any problems relating these statements to the general cultural phenomenon designated by such terms as 'cheating' or 'plagiarizing'.

In the case of Mario, we observe something different. He articulates phrases that are of the same kind as Aurélie's. He says that he does not understand, a statement that reflects the person pole of *preživanie*; and he also says, pointing to the worksheet, that 'this is dumb', reflecting something nearer the environment pole of *preživanie*. These statements themselves are intellectual rather than affective evaluations of the situation. In his case, as in that of Aurélie, there are manifestations of frustration in addition to the intellectual assessments, with attributions of the issue both to the task statement and his understanding thereof. But simultaneously, we observe in Mario's behavior persistence in the face of adversity. Spinoza understands persistence of the body and mind in their togetherness (*ad corpus et mentem simul refertur*) to continue in its own being as 'appetite' and 'desire'.[37] Mario continues on, together with the teacher even though initially there is little if any change and perhaps even aggravation in the negative emotions before the situation apparently turns around. This turnabout manifests itself in (a) positive verbal evaluations on the part of the teacher following certain {query | reply} sequences, one with a quite apparent stronger affective body, when a 'Yes' receives a very strong emphatic intonation. This is followed by another verbal assessment, 'I think you understand'. In these expressions, the turnabout in the activity manifests itself: the subjects here are both (a) integral part of the activity and (b) evaluators of its course.

Throughout the lesson fragment, Thérèse does not say much. There are a few words in an exchange with the teacher. But there are manifestations that exhibit that she knows what to do and that she is confident about what she is doing. I repeatedly point out in her as in other instances that this is not just an interpretation imposed from the outside (as this tends to occur in the constructivist tradition of research in mathematics education). Instead, what matters in the present study is how participants themselves act upon each other's behavior. Thus, Mario and Aurélie act toward Thérèse *as if* she were in the know. They do so, for example, when they look at her work (Mario, Aurélie) or copy it (Aurélie), when they ask her what she is doing and what needs to be done. These actions then are part of the internal movements that take the event into the future. The event as a whole is marked by the respective expressions of affect. More so, the event, the social phenomenon, is affective through and through.

The teacher, in the course of the exchange, also produces manifestations of frustration. She apparently does not understand Mario's lack of understanding, which persists even though she is trying to help him understand, as she reflexively articulates for him by means of a *formulating* statement. Initially there are no signs of frustration but rather of the intent to assist – she offers the query, 'what is the ques-

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[37] Spinoza, 'Ethics', 309; and Benedicti de Spinoza, *Ethica: Ordine Geometrico Demonstrata et in Quinque Partes Distincta in Quibus Agitur*, chapter 3, proposition 57, available at http://users.telenet.be/ rwmeijer/spinoza/works.htm

tion?', and then follows up by initiating an *obučenie* [teaching | learning] sequence. Participation in this event, therefore, brings about changes that are associated with manifestations of affect. That is even though affect is positively tinged at the outset it definitely is turning to a negative value. At one point, after a reply that contained something about a wedding of which there was nothing in the story, the teacher's own reply consists of an explosive 'but?' with sharply rising intonation (from 191 to 381 Hz) and an explosive exhalation of air, upon which she turns to the camera (the only time that she does so over the entire exchange). The intonated 'but' can be heard as if she is saying impatiently, 'but what are you saying?' or 'but there is nothing relating to a wedding!' The turn to the camera then may be seen as manifesting the realization of having been caught in an instance of impatience. In reply, Mario raises his hands to bury his face in the palms.

For all three students, the objective environment is the same in some sense. They are in the same lesson working on the same task. They are sitting at the same group of desks, accessing the same physical model of the story printed on their identical worksheets. Although the teacher mainly faces Mario and addresses him, there are several {query | reply} sequences initiated by the teacher where Thérèse fills the second slot; and the teacher repeatedly glances at the two girls, as if monitoring whether they are following the exchange between Mario and herself. But in the *pereživanija* of the three children, as theorized in Fig. 6.3c, this objective environment manifests itself differently. That is, their *horizons* are different, and so are their *pereživanija*. But we must not make the mistake and attribute affect to the person, for the source of the affect (as that of the intellect) is as much in the task as it is in the body of the person; and the results of the movement are found again in the environment, such as in the different forms of actions and transformations that the three students perform. Most importantly to keep in mind is Spinoza's emphasis on transitions – consistent with the dynamic perspective that Marx and Engels later developed in their materialist approach – so that we take affect as marker of transition. Thus, paraphrasing Spinoza, pleasure (pain) is the *transition* from a state of less (greater) perfection to a state of greater (less) perfection.[38]

Toward A Monist Conception of Affect and Emotion

In this chapter I make a case for considering emotions specifically and affect generally through the radical Spinozist-Marxian perspective that Vygotsky began to take near the end of his life. He never had time to develop this perspective – having seen, as he writes in his personal notes, the Promised Land from afar without ever being able to set his feet on it. In this conception of intellect and affect, the 'psychophysical problem' – today better more frequently discussed under the label of mind–body gap – is overcome because both are treated as manifestations of one and the same substance of which the individual thinking body is an expression. For

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[38] Spinoza, 'Ethics', 311.

Vygotsky, the materiality of consciousness in speech, the unity/identity of materiality (speech) and spiritual life (thinking) is the high road to a general psychology that overcomes the weakness of traditional psychology, split as it is between physiological-biological psychology and interpretive-ideal psychology.

This position implies a unity/identity of two very different, even mutually exclusive aspects. Such a statement that asserts the unity/identity of opposites may be difficult to grasp initially. It is difficult to grasp only when we try to think about the issue from the side of the manifestations. The question then will be how the intellect and affect not only can be united – parallelism does such uniting in its own way, by introducing a 'mediator' – but also can constitute an identity. As soon as we begin thinking about the issue the other way around by accepting that there is only one substance – Nature, Life – then the issue is more easily graspable. Body and mind are then the manifestations of the attributes Extension and Thought, respectively.

In the lesson fragment presented above, two forms of intellectual work can be observed. One form is concerned with the mathematical task itself, the movement from the initial task statement to its end. Here, the students produce something like a generalization: 'to find out the amount of money in the piggybank, take the number of the week, multiply it by the $3 dollars she saved each week, and at the $6 that she started with'. In addition to this intellectual side we observe part of the awareness work concerning the progress of the task toward its unknown end when the solution will have revealed itself. There are repeated articulations of a lack of understanding and queries about what is to be done.

In addition to these intellectual aspects of the work, there are affective characteristics. But these affective characteristics are not the result of the intellectual realizations. The mind would not have to inform the body and make it go through the physiological upheavals that are the inner manifestations of emotion simultaneously observed in the public sphere of joint labor. In fact, the mind would be unwise to inform the body because it can spare the upheaval and would be better of in cold consideration of the issues at hand. Instead, emotion is *another* manifestation of the whole, which we may consider in terms of *pereživanie*. Thought itself is perfused with affect, and affect is perfused with intellect – when considered from the one-substance perspective. Because affect is something very different than verbal consciousness, because it is a form of consciousness of its own, we cannot access it appropriately through the verbal route. We do not get access to affect by asking Aurélie, Mario, Thérèse, or their teacher how they feel (during the fragment, e.g., by means of wirelessly administered questionnaires) or how they felt (after the fact during interviews or by means of questionnaires). Doing so would neither overcome the mind–body problem nor lead us to an understanding of the relation of affect and intellect and, as Vygotsky states, hamper us achieving the goal 'to overcome intellectualism in the teaching on affects and to find the specific trait that distinguishes an emotional state from a purely cognitive, intellectual state of consciousness'.[39] Instead, we need to investigate the sound-words that speech produc-

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[39] Vygotsky, 'The Teaching about Emotions', 116.

es because they inherently are sensible–supersensible. Sound-words clearly are sensible, leading to the resonance of two physical, extended bodies. Sound-words also obtain a supersensible character, because they function as synecdoches of previous relations with others – 'a word', Vygotsky writes referring to the French psychologist Pierre Janet, 'was a command for others' and 'the functions of a word … were first divided and distributed among people, and then became part of the person'.[40]

In considering Vygotsky's writings on the relation of thought and language, mathematics educators tend to take up the concept of 'word-meaning'. This notion, however, is highly problematic because it is used in the intellectual ('ideal') sense alone. 'Meaning' is treated as something other than the sound-word, as something metaphysical, which is indexed by the specific words that mathematics students or teachers employ. If it were in this way, then Vygotsky would have failed in his project of overcoming the psychophysical problem at the very instant that he started it. Surely, he did better than this – as can be seen from his ample writings on the psychophysical problem and the need to overcome it. We make some inroads by beginning with the Russian term for 'word-meaning', *značenie slova*. In addition to 'meaning', *značenie* translates the English word groups (a) significance, signification, sense and (b) value, importance, weight, and magnitude. Moreover, E. Il'enkov, in addition to demarcating the term by means of quotation marks also parenthetically adds the terms 'function' and 'rôle'.[41] As soon as we consider the term *značenie slova* from the second and third sets of terms, we arrive at a pragmatic conception: in each situation, a word has a particular value, a function. Paraphrasing Ludwig Wittgenstein we may ask, for example, 'But what is the meaning of the words "model the problem"?' and then answer in the same manner, 'No such thing was in question here, only how the words "model the problem" were used'.[42] In this case, then, the dictionary senses may no longer be at issue at all, and all the functions may ride on the prosody and the affective values that manifest themselves therein.

This is exactly the case that Vygotsky himself discusses in the case of a brief text from Fyodor Dostoyevsky's *A Writer's Diary*.[43] In this text, Dostoyevsky reports overhearing the 'conversation' of six drunken workmen, which consists in six repetitions of the same word so vulgar that it is not reproduced in the text. At issue, so Vygotsky writes, is not the dictionary sense but the intonation. That is, the semantic value of the word is irrelevant and everything communicated rides on the physical properties of the sound-word, which are manifestations of the affective evaluations of something that has been said before and evaluations of these evaluations. And yet the translation states that this is 'an excellent example of the extent to which intonation facilitates subtle differentiations in the comprehension of word

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[40] Lev S. Vygotsky, 'Concrete Human Psychology', *Soviet Psychology* 27 no. 2 (1989), 57 and 61.

[41] Evald V. Ilyenkov, 'Dialectics of the Ideal', *Historical Materialism 20* no. 2 (2012), 168.

[42] Ludwig Wittgenstein, *Philosophische Untersuchungen / Philosophical Investigations* (Oxford: Blackwell, 1997), 3.

[43] Fyodor Dostoevsky, *A Writer's Diary* (Evanston: Northwestern University Press, 1994), 257–258.

meaning'.[44] A similar case was recently reported when students in a physics class participate in saying the word 'penis' ten times in a sequence.[45] But in this 'conversation', the dictionary senses of 'penis' were not at all at issue; and the whole 'game' in play was riding on the prosodic values. In a different kind of game, we might have heard, following the articulation of 'penis' the sound-word sequence 'dick', 'schlong', 'verge', 'shaft', or 'wanger', so that the goal could have been heard to be the production of synonyms. 'Meaning', therefore, used synonymously with dictionary sense, something purely ideal, makes no sense whatsoever. In the case Vygotsky discusses, the unprintable noun differently intonated *manifests* rejection, doubt, or indignation.

In the present lesson fragments, the participants do not stop to take time out, represent what someone else has said, interpret it, reflect on it, and then compose a reply. For example, in Fragment 7.2, the reply is nearly immediate and much less than the one or more seconds that it would take to interpret even something as simple as a Tetris figure composed of between one and five squares. A full reply is unfolding, beginning one-third of a second after the teacher's *Saying* has come to a close (turn 68).

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066      t:                    [why three plus six:. ]
067                 (0.34)
068      T:          because it equals to the deposit of the first week she has nine.
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In the traditional approach, it is only after the Saying has ended that a consideration of *what* has been said, the *content* of the talk could begin and, even more importantly, the *function* of the Said could be established – here, the invitation of a query that would be accepted in a reply that matches the content. If participants were to have the additional task to interpret anything in terms of its emotional qualities, then the limits of the cognitive would have been exceeded many times. In this respect, Vygotsky further notes that the 'rapid tempo of oral speech is not conducive to the development of speech activity as a complex volitional action, that is, as an action characterized by reflection, the conflict of motives, and selection'.[46] He thereby argues against the interpretive take on conversation, which is based on the assumption that interlocutors *interpret* each other's productions. What a speaker says is not (necessarily) based on the externalization of a reflection; instead speech is by and large dominated by habits. Speech constitutes a chain of rejoinders (replies, reactions). It presupposes the *con*text of speech, that is, it presupposes a common horizon. Thus, when Aurélie, Mario, and Thérèse talk among each other or with the teacher the (school, classroom) setting, task, teacher, or classmates all are part of the common context, the common ground that is presupposed in their speaking and thinking. These are part of the overall activity in which the

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[44] Vygotsky, 'Thinking and Speech', 271–272.

[45] Wolff-Michael Roth, 'Meaning and the Real Life of Language: Learning from "Pathological" Cases in Science Classrooms', *Linguistics and Education* 30 (2015): 42–55.

[46] Vygotsky, 'Thinking and Speech', 272.

task has its place. Affects and emotions are manifestations of the overall event through the lens of the {individual | collective} *pereživanija*. By thinking about the lesson fragments in this way, the events are not reduced to verbal consciousness of individuals – one of the great double sins of interpretive psychology and interpretive mathematics education – a reduction that operates on the basis of the dichotomy of mind and body and of the dichotomy of individual and collective.

In the preceding presentation of the case materials, a fist pounding on the desk or a body configuration are described by means of affect-denoting words (e.g. frustration) or a voice is denoted as plaintive. It is not that we see the configuration and then interpret it as hesitant or frustrated, but we see hesitancy and frustration even though there is no organ for perceiving these affective tonalities in a way paralleling seeing the colors red and blue of the chips in the goblets. We do not hear sound-words and then interpret them as plaintive but instead we hear the plaint as the speaking unfolds. In the way we must not reduce sound-words to dictionary sense, we must not reduce them to material expressions alone, such as these appear in the sound parameters in Fig. 7.4 and Fig. 7.7. Wittgenstein warns confusing the different issues involved, suggesting that 'a hesitant posture' or 'a timid face' cannot be described in 'purely visual' terms, just as 'hear[ing] a plaintive melody' cannot be reduced to the description of a 'perceived structure' in terms of the '"major" and "minor"' keys in music.[47]

The reverse is also the case. We do not have to assume that Aurélie, or Mario, somehow needs to be (intellectually) conscious of what is going on with them prior to expressing it in the different, affective-bodily mode. We do not feel something in our bodies and then interpret it as an emotion. We may act, in deed or word, abruptly and only then realize that we are angry, frustrated, or upset – as may be the case when Aurélie pounds her fist on the table. Everything happens together, the explosive 'doing [fais]', the high pitch, the fist pounding the desktop (Fig. 7.9). After the fact all of this is realized as, and constituted to be, a manifestation of an emotional valuation of the unfolding event in the way that it appears in the *pereživanie* of Aurélie, but also in that of the others, who witness the behavior as such. As suggested above, this recognition necessarily implicates Self and Other, for any one individual could not recognize an emotion in the behavior of others unless s/he will have taken the view of the Other on his/her own behavior and feelings. But this recognition does not require interpretation in the way this word is generally used, as a conscious elaboration of something perceived as a sign, the 'meaning' of which demands to be worked out. Instead, 'we react to the visual impression differently from someone who does not recognize it as timid (in the *full* sense of the word). – But I do *not* want to say here that we feel this reaction in our muscles and joints and that this is the sensing'.[48] A very different notion of perception and sensation is at work. That is, even though the members to a setting might quibble over the affective quality of a melody, whether or not it can be 'heard [as] a plaintive melody', they neither question the fact of the objective existence of the

[47] Wittgenstein, *Untersuchungen*, 209.

[48] Wittgenstein, *Untersuchungen*, 209.
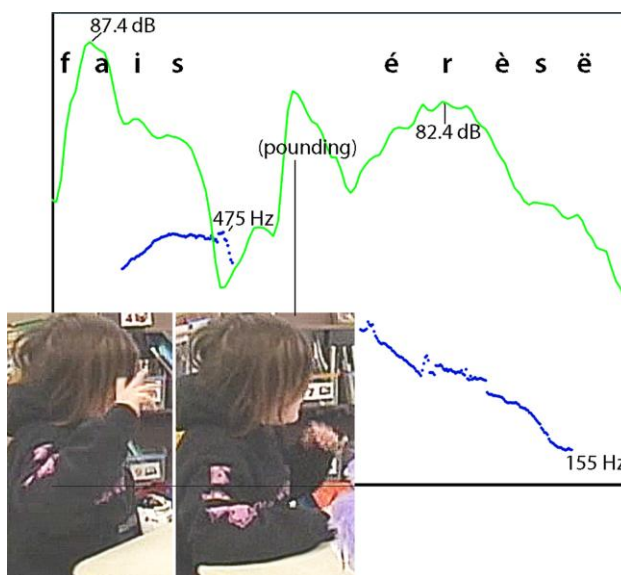
Fig. 7.9   Coincidence of language, prosody, and body movement (pounding right fist) while Aurélie comes to the end of 'qu'est-ce que tu fais Thérèse [what are you doing Thérèse]

melody (music) nor argue about differently hearing the notes (e.g. 'the melody feels [sounds] melancholic'). In the case of body movements and configurations, 'there is no ground for assuming that a man feels the facial movements that go with his expression, for example, or the alterations in his breathing that are characteristic of some emotion'.[49] This is so even if the person feels emotions as soon as her/his attention is oriented toward them, such as when asked afterwards in a conversation with an interviewer interested in understanding the role of affect and emotion in mathematics education. In each case, at issue is the phenomenon as a whole, in which the intellectual and affective are united, the hearing of a plaint, which is neither intellectual nor affective alone.

   In the preceding account of the mathematics lesson fragment, the event as a whole is emphasized. This event is not composed in some form of addition – the independent Selves of Mario, Aurélie, Thérèse, and the teacher plus whatever objective environment – or syntheses of these. Instead, there is a whole event within which parts can be identified, as articulated in the model used here (Fig. 6.3c). But these parts are not essences, not elements; these parts exist only as parts of this particular whole. These parts and their role in the event are constituted by the whole as much as they contribute to making the whole what it is. This is why we need to look at the *transactional* relevance of emotions, their participation in and characteristic of *events as a whole* – rather than some essentialized stuff that is inside individuals even if nobody can see any evidence for it (see above). Thus, if

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[49] Wittgenstein, *Untersuchungen*, 105.

Vygotsky writes that thinking is integral to the movement of substance, the unfolding event, the content of which is available only after the fact – rather than existing before and determining the actions of the agent – then he might well have eventually written he same about emoting and emotions. Because the source of emotions lies outside the individual, and because the effects of emotions lie outside again in the manner that the agent acts upon the world, we cannot get at the phenomenon if we look at the inside of the individual and even less if we look into the mind. This is precisely a Spinozist-Marxian take in the form Vygotsky might have elaborated it, where the event is considered as a whole *in its unfolding*, that is, in terms of the movement from then to the future, where a static *now* is only a fiction in the face of 'the transitiveness and open event-ness of Being', the 'actual event-ness of the once-occurrent event' of which 'aesthetic intuition is unable to apprehend … for its images or configurations are objectified'.[50] The act is whole only in the historical actuality of its being, which exists in the actual and only once-occurrent experiencing thereof, and that historical actuality is missed by statements of content or 'meaning' of the action, which may be a word or deed. Any once-occurrent event 'is actually experienced, affirmed in an emotional-volitional manner, and cognition constitutes merely a moment in this experiencing-affirming'.[51] Vygotsky is concerned with the event-ness of life within which thinking plays a part. *Pereživanie,* the relation of the person to her environment, is precisely that lens that includes how the person becomes aware of and emotionally relates to the event concerned.

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[50] Bakhtin, *Philosophy*, 1.
[51] Bakhtin, *Philosophy*, 13.

8

Obučenie – Teaching | Learning as Event

In his works, Vygotsky employs the notion *obučenie*. It corresponds to the English terms 'teaching', 'instruction', 'training', 'education', 'course', and 'learning'. Thus, when the term is translated as 'teaching' or as 'learning', it does not render what the original Russian text really is about. That is, the Russian term 'refers to a double-sided process, one side of which does indeed refer to learning (a change in the psychological processes and knowledge of the child), but the other of which refers to the organization of the environment by the adult', generally an institutionally designated teacher.'[1] Though not having been taken up widely, the compound notions 'teaching/learning' and 'complex of teaching/learning' have been proposed and used in studies simultaneously focusing on curriculum/instruction and the forms of learning that arises from such organized situations.[2] *Obučenie* thereby takes into account that classroom events cannot be reduced to instruction or pre-specified curriculum outcomes or to creative learning processes entirely internal to the students. It has recently been suggested that current work concerning the notion leaves us with a 'more adequate, but still incompletely articulated, way of thinking about the dynamics of development involving such ideas as the social situation of development and neoformations, as well as the differences between formal instructional settings and the many other forms of activity organized for and by children'.[3] In this chapter, I take a Spinozist-Marxian stance to develop related scholarship; in this stance, relations are conceived as transactional and, therefore, as

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[1] Michael Cole, 'The Perils of Translation: A First Step in Reconsidering Vygotsky's Theory of Development in Relation to Formal Education', *Mind, Culture and Activity* 16 (2009), 292. See also Wolff-Michael Roth, 'The Role of Soci(et)al relations in a Technology-rich Teaching | Learning Setting: The Case of Professional Development of Airline Pilots', *Learning, Culture and Social Interaction* 7 (2015): 43–58.

[2] See, Denis Newman, Peg Griffin, and Michael Cole, *The Construction Zone: Working for Cognitive Change in School* (Cambridge: Cambridge University Press, 1989), 29; and editorial comment in Lev S. Vygotsky, and Alexander Luria, 'Tool and Symbol I Child Development', in *The Vygotsky Reader* eds. René van der Veer and Jaan Valsiner (Oxford: Blackwell, 1994), 114.

[3] Cole, 'Perils', 295.

being irreducible to teacher and student, curriculum and learning, or individual and social.

In the mathematics education literature, the use of the term *obučenie* was proposed as a way of theorizing teaching and learning 'around the shared work of the students and the teacher in a space of joint action, in the course of which the students attend to and become familiar with historically and culturally constituted forms of thinking'.[4] Things in the environment, such as three-dimensional objects, have been used as part of cultural practice in the past and, thus, have a cultural history; but because they are now part of the environment they shape, from Vygotsky's Spinozist-Marxian perspective, the behavior of individual and collective alike. Thus for example, although children might find cubes, cylinders, and spheres outside of school as well, by their special appearance in the lesson called 'geometry', these objects become associated with performances in this class, including talk about edges, vertices, and faces. It is in the joint labor of students and teachers that particular forms of behavior first emerge collectively – from the perspective of the child – a form that it later will perform on her own (see chapter 4), whereas it emerges *again* as collectively from the perspective of the teacher for whom the behavior already had been fused into one. Most importantly, *obučenie* therefore allows 'individual consciousness to emerge as a concrete realization of historical collective consciousness' in classroom-based human relations that 'do not constitute negotiations of significations ("meanings"), the bargaining of knowledge stuff that epistemic subjects exchange'.[5]

A major difficult arises for Western scholars from the ways in which *obučenie* has been translated, in part because there is no equivalent English word. For example, in the foreword to *Myšlenie i reč'* [*Thinking and Speech*], Vygotsky uses the expression 'psychological theory of *obučenie*', which in English is rendered as 'educational theory' whereas in German it becomes 'psychological theory of instruction [psychologische Unterrichtstheorie]'.[6] Chapter 6 of *Mind in Society* is entitled 'Interaction between Learning and Development', where 'learning' has been used to render the same Russian *obučenie.*[7] However, such a translation does not make sense, especially in the context of the Russian interest in the relationship of *obučenie* and individual development. As soon as the translation is reconsidered, other aspects of Vygotsky's work have to be reconsidered, too. For example, once we consider the verb 'to develop' as both transitive (i.e. we develop particular perspectives or theories) and intransitive (i.e. we develop, grow, change), a Spinozist-

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[4] Luis Radford and Wolff-Michael Roth, 'Intercorporeality and Ethical Commitment: An Activity Perspective on Classroom Interaction', *Educational Studies in Mathematics* 77 (2011), 237.

[5] Radford and Roth, 'Intercorporeality', 243.

[6] See Lev Vygotsky, 'Thinking and Speech' in *The Collected Works of L. S. Vygotsky: Volume 1: Problems of General Psychology* (New York: Plenum Press, 1987), 39; Lev S. Vygotskij, *Myšlenie i reč' : psixologičeskie issledovanija* (Moscow: Gosudarstvennoe social'noèskonomičeskoe isdatel'stvo, 1934), 1; and Lev S. Vygotskij, *Denken und Sprechen* (Weinheim: Beltz Verlag, 2002), 38.

[7] Michael Cole, one of the editors of *Mind in* Society, later admitted to this as a mistake. See Cole, 'Perils', 292; Lev S. Vygotsky, *Mind in Society: The Development of Higher Psychological Processes* (Cambridge: Harvard University Press, 1978), 79.

Marxian approach becomes suitable. In this approach, we are subject and subjected to conditions that we, qua subjects, contribute to producing.

In this chapter, I present and unfold a Spinozist-Marxian approach in which the fundamental unit is the {teaching | learning} unity/identity.[8] This does not mean that we equate teaching, what a teacher does considered independent of the student, with learning, what a student does considered independent of the teacher. Instead, in considering this unity/identity of {teaching | learning}, teaching cannot be understood independent of learning, and learning cannot be understood independent of teaching. Furthermore, {teaching | learning} means that in any classroom situation, 'teachers' may not only teach but in fact learn; and institutionally identified 'learners' may not only learn but in fact contribute to the teaching of teachers. When a student does not exhibit some form of behavior anticipated in the curriculum, then it is not just the student who fails, but the teacher also fails – because *obučenie* denotes their joint labor. The curriculum fails as well, because it was defined independently of any particular student's current developmental situation. *Obučenie* is the manifestation of a particular form of life-activity where participants 'are more than *in* the world' because the world itself is the result of their relation to each other, their joint interventions, transformations, dreaming, apprehensions, and hoping.[9] And, as mathematics educators have pointed out, in such cases it is the *obučenie* process itself that would be failing[10] – not unlike in the discourse whereby it is the school that fails certain children. In this chapter, I exemplify and develop this notion with mathematics classroom events that exhibit to the teacher the insufficiency of her current knowing, which comes to expand in the context of classroom praxis where she figures out ways of overcoming student difficulties.

Obučenie – A Drama Perspective on Classroom Life

When Thomas (t) comes forward, a small drama plays itself out, where the stage is the front of the classroom with various props – including the poster the teachers (Mrs. T. and Mrs. W.) had prepared and added to (Fig. 8.1) and the various three-dimensional objects on the tray underneath the chalkboard – and the remaining people present (other students, researchers) constituting the audience. The frag-

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[8] Michael Fried recognizes the dialectical nature of one teaching and learning event. Anna Stetsenko uses the expression teaching-learning, which does not sufficiently emphasize the dialectical nature that is to be developed in the present volume. See Michael Fried, 'A Cultuural-Historical Perspective on Mathematics Teaching and Learning, by Wolff-Michael Roth & Luis Radford', *Mathematical Thinking and Learning* 15 (2013): 83–88; and Anna Stetsenko, 'Teaching-Learning and Development as Activist Projects of Historical Becoming: Expanding Vygotsky's Approach to Pedagogy', *Pedagogies: An International Journal* 5 (2010): 6–16.

[9] Luis Radford, 'Education and the Illusions of Emancipation', *Educational Studies of Mathematics* 80 (2012), 110.

[10] Radford and Roth, 'Intercorporeality', 239.
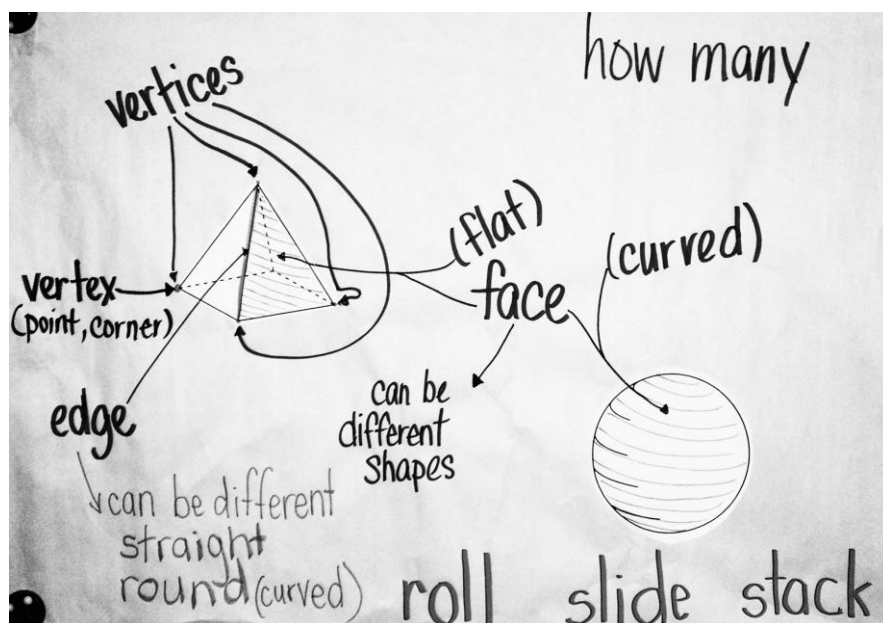
Fig. 8.1 The poster teachers prepared (top part) and some of the 'outcomes' of the lessons that were added after lesson fragments such as the one with Thomas: 'edge → can be different, straight, round (curved)'

ment picks up with the lines that featured in chapter 4, at a point in time when we observe a failed solicitation to state the feel or type of an edge {turn 22 | turn 23}. We may gloss what has happened so far by saying that the two teachers made several attempts at soliciting a statement concerning the feel or type of edge associated with the cube in Mrs. W.'s hands; and, after a cylinder is made part of the accented visible, a statement is solicited about the difference between the edges of the two objects. Eventually, a statement comes forth, though, as shown in the reply 'if you just did one of them', it was about a multiplicity of something of which the sought-after statement was to address 'just one'. It is apparent that the first solicitations have been unsuccessful until a reply comes forth that then constitutes the departure for the continuation of the talk. The *obučenie* drama exhibits its true nature, because success is never guaranteed and it may fail at any instance.

Fragment 8.1a

21 W: but your its part of one verticee; put that finger on that verticee. (0.8) your finger (0.3) point (0.6) one verticee (0.5) and r*un*:it along that edge to the other verticee (0.8) okay? (1.2) whats that feel like; what kind of edge is that. ((*Mrs. W. runs his finger along the edge*))

22 (4.2) ((*scratches his head*)) ((*Mrs. W. gets the cylinder*))

23 W: kay, (.) where (0.6) all right th*is* time; (0.7) put your
      finger *out* ((*Thomas puts finger out, Mrs. W takes it
      and places it on the edge of the cylinder*))

24   (1.1)

25 W: what does th*at* one feel like. ((*moves his finger along
      the edge*))

26   (0.9)

27 t: it feels like um;

28   (4.3) ((*Mrs. W. moves his finger repeatedly around the circumference*))
      ((*Thomas has questioning look*))

29 T: does it feel the s*ame* or does it feel different;

30   (0.3)

31 t: feel d*iff*erent.

32   (0.9)

33 T: what is different about those two edges.

34   (0.8)

35 t: because um this one is round

      and this one is ap (0.5) isa square ((*moves finger
      around the four edges of the top square*))

36   (1.6) ((*Thomas looks up to Mrs. W.*))

37 W: if you j*us*t did *one* of them ((*moves finger up and
      down an edge of the cube*))
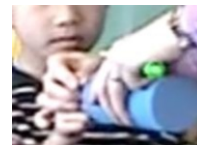
We might ask at this point, why does it take so many solicitations until that point? As this fragment part unfolds, in the give and take between the two participants, solicitations are tested in the apparent face of an uncertainty about which one will work. But, teachers often find themselves in situations where they go through a series of questions until some intended or acceptable reply is given only to begin a sequence with another student or group using the ultimately successful question after having sensed that the new situation shares similarities with the old. That is, teachers learn in events, such as the one unfolding here before our eyes, where the work of {querying | replying} involves finding a statement that does the trick of bring about a reply to a query. When teachers will have learned posing the right kind of queries, their learning can be traced back to events such as the present one, where it is precisely Thomas who, in replying or being silent, provides Mrs. W. with feedback about the appropriateness of the query with respect to the in-

tended or situationally acceptable reply.[11] But out of the same event may arise new forms of behavior on the part of Thomas, which, when we trace them back, will first have appeared as the relation with Mrs. W. (or Mrs. T.). Instead of thinking about and theorizing such events as 'teaching' or 'learning', we are thus better off focusing on the joint work accomplished. To keep our focus specific, it is useful to employ the term *obučenie* as a category for events typical of lessons in which schooling is concretely realized. It orients us to the fact that '*obučenie* requires togethering, the committed engagement in joint activity with a common ideal/material object, as there is only hope for instruction when the teacher speaks in a way that is anticipated to be comprehensible to the student, and when the student speaks in a way that is anticipated to be comprehensible to the teacher'.[12] But comprehensibility has to be worked out in joint labor, through speaking and listening to the effect speaking brings about in the talk of the other.

   The preceding fragment shows that the joint work of *obučenie* may consists of {query | reply | evaluation} sequences that have come to be referred to by the acronym *IRE*, standing for initiation, reply, and evaluation. The evaluation is part even though it may not be seen as immediately apparent. Consider the function of turn 33 in the sequence {turn 29 | turn 31 | turn 33}. On the surface, turn 33 offers up a query, as per the use of an interrogative 'what'; it follows another turn at talk on the part of Mrs. T., which also has the grammatical structure of a question ('does it …') and the typical intonation (pitch contour) that rises toward the end. The second query, however, is not just a reiteration or an elaboration. It also marks that whatever has been said to be insufficient. We have the same kind of relation in the sequence {turn 33 | turn 35 | turn 37}, where turn 37 not only can be heard as an invitation to limit the statement to 'just one of the edges' but also as an evaluation that whatever preceded did not produce what was asked for.

   Such sequences are social phenomena. They are typical of educational situations, frequently formal schooling, but can also be found in other types of situations such as nature centers. In fact, they are typical of situations where someone is constituted to be in the know when compared to another or others in the situation. Unlike situations where we ask a genuine question – directions to get to our intended destination, time if we do not have a watch, or price of a commodity if it is not displayed – Mrs. W. already knows the answer. At least, as the unfolding event shows, what she says remains uncontested and, therefore, counts as legitimate – even though, for example, we hear, in turn 22, an erroneous form of the word vertex based on the omission of the 's' from the plural form 'vertices' resulting in /ˈvɜrtɪ siː/. It is not that Mrs. W. knows and therefore gets to evaluate that power arises from knowledge. Instead, 'power' (being in or having power) and 'knowledge' (being in the know) are the byproducts of transactional relations. Thus, as soon as someone would be contesting a statement that came out of Mrs. W.'s mouth, and as soon as the ensuing debate shows that the statement was

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[11] See, for example, Wolff-Michael Roth, 'Science Teaching as Knowledgeability: A Case Study of Knowing and Learning during Coteaching', *Science Education* 82 (1998): 357–377.

[12] Radford and Roth, 'Intercorporeality', 242.

wrong, being in the know would be attributable to someone else. This is precisely what was observed between an undergraduate student and his physics professor talking about a graph, where there is an ongoing struggle over who is in the know about the particular graph and who is less so.[13]

Another part of the work performed concerns the production and removal of ambiguities as to the topic of talk. Thus, those in the know hear Mrs. W. and Mrs. T. 'ask questions' about edges, first pertaining to those of the «cube» and then about the difference between those found on the «cube» and on the «cylinder».[14] Thomas says, moving the finger around the «cylinder» and top face of the «cube», respectively, 'this one is round and this one is a square'. The statement is taken up to be about four edges, as can be seen from the offered invitation to 'do just one of them' and the coincident movement of the finger along one of the edges of the cube. However, as supported by a later statement, Thomas might orient the teacher toward the faces, one of which is round like a circle that the finger following the circumference also describes, whereas the other is a square, again marked by following the circumference of the top face. That is, *obučenie* has emergent qualities so that none of the participants can anticipate with any certainty future events, though very experienced teachers may frequently come close to initiating successful {query | reply} sequences without requiring as many turns as observed above.[15]

Returning to the lesson fragment, in referring to all four edges of the top square of the cube, a new situation will have arisen, as shown in the {invitation | acceptance} sequence as part of which first Mrs. W. and then Thomas move a finger up and down one of the vertical edges. But again, a turn sequence that determines a reply – i.e. 'feels like … triangle' – as inappropriate, when a pyramid is offered up together with a constative statement: 'that would be a triangle'. Another verbal statement states an instruction or interdiction: '[we are] not using shape words' (turn 46). There is another long pause, which is then treated as if a reply will unlikely come forth and other students are called upon to provide whatever was to have been said on the part of Thomas but what he actually did not say.

Fragment 8.1b

38   (0.8) ((*Thomas nods*))
39 W: what would it (0.6) f::eel like.
40 t: feel like um this one (0.7) ((*moves finger up and down the edge*)) is um (.) triangle,

41   (1.8) ((*Mrs. W. gets a pyramid*))

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[13] Wolff-Michael Roth, and David Middleton, 'The Making of Asymmetries of Knowing, Identity, and Accountability in the Sequential Organization of Graph Interpretation', *Cultural Studies of Science Education* 1 (2006): 11–81.

[14] The chevrons are used to mark that the object itself is referenced rather than the word and its 'meaning'.

[15] See also Newman et al., *Construction Zone*, 11–13.

42 W: °okay° that would be the triangle.
43   (0.8)
44 t: triangle and (0.6) and this one feels like um; (1.1)
         a triangle ((*moves around the triangular face of*
         the pyramid))

45   (1.0)
46 W: right; not using a shape word. (0.2) so we are not
         using a triangle circle or square.
47   (1.9) ((*Mrs. W. moves finger up and down the*
         edge of the cube))
48 W: can someone help us out?

   In one way of thinking about the unfolding event so far, the *obučenie* relation
has not produced what is to be the intended outcome – which mathematics educa-
tors, already being in the know, might immediately see from the unfolding talk and
which the subsequent turns indeed confirm as such. Not only has *obučenie* been an
unsuccessful event in this way of seeing, but also it is abandoned at least momen-
tarily. In popular discourse expressed, Mrs. W. may be said to have 'abandoned'
instructing Thomas and now is turning to others to produce through their perfor-
mances, whatever is required so that the lesson then can continue.

   Looking at the reply turn closely, 'this one feels like … a triangle', we note that
from a particular point of view, it is not at all unreasonable. At the edge, two faces
meet, like the two parts of a roof, forming a triangular shape. That is, if the finger
moves over the edge orthogonally to it, then indeed it 'feels like a triangle'; but if
the sought-after reply concerns the edge considered longitudinally, then 'triangle'
apparently is inappropriate. In the case of the pyramid, the term 'triangle' also is
appropriate for describing the shape of the face. We do not know what form of
thinking or perception manifests itself in this statement, but in its articulation, it is
a public issue. From the instructional side of *obučenie*, the issue is treated as a
problem of using a shape word, triangle, where something else is to be used that is
not a shape word.

   The fragment that follows may be glossed as 'eliciting and providing the
sought-for word that would fit the bill of the formal curriculum, such as common
attributes of three-dimensional objects'. There is an {elicitation | reply | affirma-
tion/evaluation} sequence that constitutes 'straight' as the answer intended, at least
on the part of the teacher. An articulation is then offered, which might be heard as
the summary of what is to be taken away from the events that have unfolded before
everyone's eyes: the edge of a cylinder is round whereas the edges of a cube are
straight. There is what we can hear as a query (turn 55) – because of the grammati-
cal structure – but no reply. We never do find out whether in fact anyone agrees or
whether there are dissonant voices. The intonation that comes with the declarative
statement exemplifying a straight and round edge may not encourage the produc-
tion of a dissonant reply.

Fragment 8.1c

49   (0.7)
50 W: he said th*is* one is ((*moves finger around the circumference of cylinder*))
51   (1.7) ((*moves to look at his face*))
52   however what we said; *roun*d (1.1) what would this one b*e*:? ((*feels edge of cube*)) (0.4) al*ic*ia.
53 A: straight.
54   (1.0)
55 W: can we agree?
56   (0.4)
57   this one is straight ((*moves finger along edge of cube*)) (0.83) th*is* (.) edge ((*moves around cylinder*)) (.) is round.
58   (0.4)

In this fragment, therefore, two adjectives that describe edges come to be produced – not merely *in* relation but *as* the relation, embodied in the very materiality of their joint talk. The joint labor includes solicitation, reply, and evaluation. Both adjectives are verbalized by students, but, as described in chapter 3, not independent of the teacher, *for* whose ears the replies have been produced, and who has solicited them. That the two adjectives fit within the intended curriculum will become apparent when Mrs. W., once Thomas heads back to his seat, writes them on the poster where three-dimensional objects and their parts are publicly displayed (Fig. 8.1).

The fragment does not come to a close at this point, even though its 'take-home message' has been stated. We witness this when the second teacher, Mrs. T. addresses Thomas by naming him as the intended recipient, 'Would you agree with that Thomas?', and the sought-for agreement is provided. The event might have come to a close with this affirmation: but it has not. There is what we may gloss as 'an assessment'. The transcription shows that more drama unfolds when a reply is treated as inappropriate. Thus, the invitation to show a straight edge is paired with an index finger that moves in a way that is seen and treated as 'going right through the face'. Mrs. W. grabs Thomas' right hand with its index sticking out and together they move through the center of the face again, which falls together with the statement 'we are going right through the face', which is followed by a second joint movement along an edge, which is accompanied by the verbal statement 'here is our straight edge' (turn 68). The fragment comes to an end with an accepted invitation to 'go through the round edge' of the offered cylinder, which finds its evaluation in a 'good for you', which is then followed by the accepted invitation to return to the seat.

Fragment 8.1d

59 T: would you agree with that thomas?
60   (0.4)
61 t: .h uh yea.
62   (0.4)

63 W: so which one is the straight (0.80) edge.
64   (0.4)
65 t: this one ((*moves finger down the face of the
         cube*))
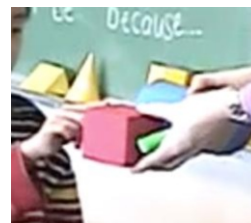
66 W: is it down h*ere*? showm– put your finger down straight edge.
67   (1.5) ((*Thomas moves finger down the middle of the cube face Fig*))
68   now when y*ou*=re ((*captures his hand*))

68 W: doing this we are going right through the face.
         (0.68) here is our str*ai*ght edge ((*moves his fin-
         ger along the edge of the cube*)) now go
         through the r*oun*d edge ((*Thomas puts finger on
         the circular edge of cylinder, moves it around*))

69 t: °°this one.°°
70 W: good for you.

At that point, Mrs. W. turns to the chalkboard where the poster hangs that she has prepared together with Mrs. T., adds an arrow downward from 'edge' into the empty space and writes, 'can be different, straight, round (curved)' (Fig. 8.1).

In this fragment, we observe a social relation – which itself requires it to be material or we would be unable to observe it. The sound-word *is* that relation, resonating simultaneously in the mouth of one and the ear of the other. The relation also manifests itself and comes to life through the hands, where Mrs. W. takes Thomas' hand with the outstretched index finger to move it along the edges of a cube and a cylinder. The 'hand work' is associated with the talk. In this aspect, the fragment is reminiscent of the work that the staff at A. Meschcheryakov's clinic accomplished with the deaf-blind children, where the former actively led the children's hands to get these to participate in the form of behavior. Initially, the behavior is collective, and later it is done individually – remaining, in its recognizability as a particular kind of behavior, social through and through.

There is also what we can hear as the work of giving and following instructions – where it is still open whether the latter is successful or not. Here, too, there is a constitutive relation at work, for something like 'put that finger on that verticee' or 'run it along the edge to the other verticee' (turn 21) is nothing but an account or plan of an action, here articulated verbally. This action, however, is to be considered as an action in the world rather than as an action according to the paradigm of the text, that is, an action reduced to a description. The relationship we are looking

at here is that between the account (thought) and the real doing (body), the real work of what the account specifies: 'doing [put that finger on that verticee]' or 'doing [run it along the edge to the other verticee]'. The work is accomplished jointly, for Mrs. W. provides the account and Thomas does the action. As seen in Fragment 8.1d, the two are not easily aligned even in what may appear to be the trivial case of running a finger along an edge.

There are studies that have shown the inherent openness and underdetermined nature of real actions by the accounts even when the most highly educated individuals of our society are involved, such as engineers and scientists.[16] In fact, Spinoza already noted that '*the body cannot determine the mind to think, nor can the mind determine the body to motion or rest, or to anything else (if there is anything else)*'.[17] Thus, rather than saying that Thomas has some kind of deficit, events such as this one help us understand the tremendous work underlying all those instances where a second-grade student already comes to do the cultural movements that the teacher-provided account specifies. The joint *obučenie* work of Mrs. W. and Thomas is concerned with producing an alignment between the verbal accounts and what Thomas does. That work involves feedback, evaluation, which marks a particular movement as unsuccessful, such as when Thomas points to a vertex when those in the know can hear the teacher ask to follow along one of the edges. An evaluation may also be produced in the form of a description of what has been done so that it can be heard (though it does not *have to be*) as contrasting what should have been done – as in the case where there is an invitation to identify the 'straight edge', which comes to be associated with a finger moving through the center of the face (turn 63 | turn 65).

The language of the preceding paragraphs orients us to the *transactional* nature of knowing and the relation between acting in the world and descriptions thereof. Within a strictly Vygotskian discourse, the 'account' is what Vygotsky refers to as doubled experience, the way in which primary experience appears in and to consciousness. The account not only instructs Thomas to do something, but also, when the doing is successful and therefore recognizable as a cultural feature, constitutes a description. It is therefore a manner of making that movement present again. That is, the movement is (has been) present twice: as bodily movement and as represented in the verbal description (mind). These instances of the doubling of experience occur by means of the work of *formulating*. Here, a physical movement coincides with a verbal statement (itself physical through and through) that denotes the movement or something about the movement. Initially, the two are doing this formulated work together: 'doing [going right through the face]', followed by 'doing [(going through) our straight edge]'. The behavior is joint, as both hands do the movement, where Thomas' hand is no less active – in letting itself be taken – than

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[16] See, for example, Lucy Suchman, *Human-Machine Reconfigurations: Plans and Situated Actions* (Cambridge: Cambridge University Press, 2007) and Wolff-Michael Roth, 'Radical Uncertainty in Scientific Discovery Work', *Science, Technology & Human Values* 34: 313–336.

[17] Baruch Spinoza, 'Ethics', in *Complete Works,* trans. Samuel Shirley (Indianapolis: Hackett Publishing, 2002), 279.

Mrs. W.'s hand taking it. Their joint work, too, is articulated as such, in '*we* are going' and '*our* straight edge'.

The words are to be taken not only in terms of purely ideal 'meanings'. As sounds, they are part of the material setting. They are exchanged, thereby producing the relation. Moreover, the words also stand for the situation as a whole. This function, standing-for, constitutes the ideal, supersensible side of the words. But this standing-for is realized pragmatically, that is, objectively, in the uses of the words and the changes that these uses bring about in and to the situation. The ideal character of the words, their 'meanings', thus is as material as anything else in the situation. The ideal exists in a continuous, 'reciprocating movement of the two opposing "metamorphoses" – forms of activity and forms of things in their dialectically contradictory mutual transformations'.[18] The ideal, thus, is a form of a thing, here the word, outside of the thing, materialized in and as relation between two people.

The database does not include a subsequent assessment of Thomas. We therefore do not have evidence of any 'outcomes' of the present *obučenie* event. But if at some time later we had observed Thomas correctly identifying the edge of a cube and to characterize it as straight, then we might have been able to backtrack the behavior to this class and to the particular exchanges presented here. For Thomas, the work and behavior described as 'doing [following the (straight, round) edge]' then would have been accomplished for a first time in, and more importantly *as*, the relation with Mrs. W. We would know that we have observed another instance of *sociogenesis* (see chapter 4).

In this extended fragment, we not only observe the work of coordinating an action with its account but also the production of the persons involved. Here, the Vygotskian perspective differs substantially from the (radical, social) constructivist or enactivist views. In these views, the individual is 'informationally closed'. This statement implies that the individual is an *autopoietic* project, even though those with social constructivist or sociocultural bents suggest that a social construction might precede the individual construction (see Fig. 1.2b). Thus, 'given that [the organism] can, nevertheless, produce descriptions, i.e., concepts, conceptual structures, theories, and eventually a picture of its world, it is clear that it can do this only by using building blocks which it has gleaned through some process of abstraction from the domain of its own experience'.[19] In contrast, in Vygotsky's Marxist take, the individuals are the byproducts of their joint labor. Thus, there is no evidence that Mrs. W. or Thomas actually engaged in the production of identity, that they somehow 'positioned' themselves or one another. Instead, the protocol of their transactional work shows that the two are oriented to the production of *obučenie*, as part of which Thomas comes to correctly do what a series of instructions describe. Mrs. W., in the course of a number of different ways of acting, comes to find one that actually gets Thomas to act until such a point that the sub-

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[18] Evald V. Ilyenkov, 'Dialectics of the Ideal', *Historical Materialism 20* no. 2 (2012), 192.

[19] Ernst von Glasersfeld, 'Die Unterscheidung des Beobachters. Versuch einer Auslegung', in *Zur Biologie der Kognition* eds. Volker Riegas and Christian Vetter (Frankfurt: Suhrkamp, 1990), 281.

sequent evaluation is accepting it ('good for you'). Now, the lesson can move on. These events, in their course, allow Mrs. W. to emerge as a teacher, even though she is learning herself, and they allow Thomas to be a learner, here coming to co-ordinate an instruction/description with a body movement. It is precisely because of Thomas' failure to succeed that Mrs. W. learns to ask the right question and, therefore, becomes a better teacher. 'Teacher' and 'learner' therefore are outcomes of the joint labor that coincide with the production of a 'prescribed learning out-come' and the associated 'suggested achievement indicator' (see below and Fig. 8.2). These terms should not be conflated with the institutional positions they take, which are referred to by the same words ('teacher', 'student'). Thus, 'it is apparent here that individuals undoubtedly make *one another*, physically and mentally, but do not make themselves … in the sense of the "unique", of the "made" man'.[20] In the dramatic event, Mrs. W. becomes teacher, and Thomas becomes learner; but Mrs. W. also is a learner, because, not taking that approach that will take them immediately to Thomas' articulation of the difference between round and straight edges, she has to continue participating in the joint labor to find out a strategy that works. And Mrs. W. is taught by Life, part of which is Thomas. It is the *event* – i.e. *obučenie* – that produces both. 'The real mental richness of the individual', Marx and Engels note, 'depends entirely on the richness of his real relations'.[21]

The notion *obučenie* includes openness and indeterminacy. Thus, in the lesson fragment success is never guaranteed, in which case we cannot speak of teaching (instruction) or learning to have been achieved. If teaching (instruction) and learn-ing had not occurred, then we could not say that there was teaching and learning, or that there were a teacher and a learner present. It might be said that the teacher and the student 'had failed', but in this case we would have made the ontological commitment to autonomous Selves that antecede the event where they come to *inter*act (Fig. 1.2b). The transactional perspective is different in that the nature of the parts – the participants, etymologically speaking, literally are taking (capĕre) (a) part (pars, partis) – cannot be specified independent of the whole event under consideration (Fig. 1.2c). One recent cultural-historical analysis of the events in a fourth-grade mathematics lesson describes how the institutionally designated teacher and student fail, and are failed, to produce successful outcomes (see chap-ter 7).[22] They do not even get to initiate a task-oriented joint work, leaving the stu-dent (Aurélie) frustrated, saying that she does not understand and will never under-stand, and leaving the teacher with a failed opportunity to instruct.

In this way of thinking about the production of the person, the real material pro-cess of production, human praxis, is used to articulate and explain the evolution of consciousness, knowing, skill, etc. The investigation always remains on the real ground of the real life of society and the individual. It explains what people do, praxis, not on the basis of ideas – i.e., plans, beliefs, intentions, or motivations –

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[20] Marx and Engels, *Werke Band 3*, 37.

[21] Marx and Engels, *Werke Band 3*, 37.

[22] See Wolff-Michael Roth and Margaret Walshaw, 'Rethinking Affect in Education from a Societal-Historical Perspective: The Case of Mathematics Anxiety', *Mind, Culture and Activity* 22 (2015): 217–232.

but explains the formation of ideas and consciousness out of material praxis, here *obučenie*. Thus, in the real praxis of their mathematics lesson, Thomas and Mrs. W. make the conditions of *obučenie* as much as the jointly produced conditions make them. We usefully think about *obučenie* in terms of drama, especially one with uncertain outcomes. Just as a major argument or fight in a couple may arise from an innocuous comment, the direct and indirect outcomes of *obučenie* inherently are dramatic in nature.

Teaching ≠ Learning

In this chapter, I present an extended analysis of a mathematics lesson fragment. The analysis does not presuppose that whatever was planned on the part of the two teachers conducting the lesson specifically and the geometry curriculum more generally actually translated into student learning. 'Learning' tends to be assumed in expressions such as 'students were taught …' Instead, the term *obučenie* is used to denote a social phenomenon in which teaching and learning may be found, but without institutional position-based a priori commitments to who learns or who teaches. In fact, institutionally designated 'teachers' learn in teaching, and they do so in their relations with institutionally designated 'students'; and institutionally designated students ('learners') may in fact learn and expand their potential to act despite of teachers' specific doings. Rather than focusing on what teachers and students do separately, which may but does not have to be 'teaching' and 'learning', we are better off considering lessons in terms of the drama of social life, where participants accountably produce the ordered and orderly forms of their activities and associate outcomes.

Mathematics Instruction as Plan

The notion *obučenie* should not suggest that teaching is identified with learning. One cultural-historical analysis of schooling exhibits the standard practice of such a reduction in schooling practices, a reduction that has been termed *Lehrlernen*.[23] What students learn, are to learn, and are held accountable for having learned in a lesson is equated to whatever teachers plan. In this way, learning comes to be articulated as a product of teaching, and the quality of teaching is assessed in terms of the scores that students achieve on standardized examinations. Such examinations, therefore, become high stakes not only for students but also for teachers. The assumption that learning can be planned is inherent in government- or school-

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[23] Klaus Holzkamp, 'The Fiction of Learning as Administratively Plannable', in *Psychology from the Standpoint of the Subject: Selected Writings of Klaus Holzkamp* eds. Ernst Schraube and Ute Osterkamp (Basingstoke: Palgrave Macmillan, 2013), 115–132.

SHAPE AND SPACE (3-D OBJECTS AND 2-D SHAPES)

General Outcome: Describe the characteristics of 3-D objects and 2-D shapes, and analyze the relationships among them

Prescribed Learning Outcomes	Suggested Achievement Indicators
It is expected that students will:	*The following set of indicators may be used to assess student achievement for each corresponding prescribed learning outcome.* *Students who have fully met the prescribed learning outcome are able to:*
C7 describe, compare, and construct 3-D objects, including – cubes – spheres – cones – cylinders – pyramids	❑ sort a given set of 3-D objects and explain the sorting rule ❑ identify common attributes of cubes, spheres, cones, cylinders, and pyramids from given sets of the same 3-D objects ❑ identify and describe given 3-D objects with different orientations ❑ create and describe a representation of a given 3-D object using materials such as modeling clay ❑ identify examples of cubes spheres, cones, cylinders, and pyramids found in the environment

Fig. 8.2 Some of the 'Prescribed Learning Outcomes' and 'Suggested Achievement Indicators for second grade according to the British Columbia mathematics curriculum

system-issued curricula, such as the *Integrated Resource Package* from the British Columbia Ministry of Education, which specifies the 'prescribed learning outcomes' and the associated 'suggested achievement indicators'. For the second grade that participated in the research featured in the preceding lesson fragment, the outcomes and achievement indicators include those for the topic of shape and space generally and those relevant to 3-D objects specifically (e.g. Fig. 8.2).[1] An inspection of the achievement indicators shows that the tasks represented in this book – sorting 3-D objects, create and describe a representation of a given 3-D object using materials such as modeling clay, and, specifically to this chapter, identify common attributes of the listed 3-D objects – all arise out of government prescriptions. It is apparent from the very discourse used in the document that learning outcomes are specified independent of the children that are actually in the classrooms and their specific needs. Students, such as Thomas, 'are expected' to exhibit certain levels of achievement independently of where they are from a developmental perspective and independently of their current horizon and any potential that he has for changing and being changed in the course of his participation in the unit. The problem is acerbated in school systems where not only the students within a class are to be aligned in terms of their learning and development but indeed all students at the same grade level in a school and even within the district.

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[1] Ministry of Education, Province of British Columbia, *Mathematics K to 7: Integrated Resource Package 2007* (Victoria: Author, 20007), 110.

We also see from the poster that the teachers created (Fig. 8.2) that those specifications drive the curriculum even under such special circumstances where they teach together with a university professor and even when they themselves stand out because of their involvement in provincial curriculum initiatives. Even though there are struggles and contradictions in the joint work performed by the teachers and Thomas, the actual outcome – as apparent in Mrs. W.'s public addition to the poster and again independent of whether it pertains at all to Thomas – comes to be described, as per the words that Mrs. W. added to the poster, in the terms of what is prescribed and expected. That is, characteristic of 3-D objects are their 'edges', and these may be 'straight' or 'round' (Fig. 8.2).

Such documents may fruitfully be considered in terms of plans or accounts with respect to situated actions. Any aspect of the planned curriculum may then be considered in relation to the actual doing – not in a normative but in a descriptive manner – such that the relationship between actual happenings and prescriptive account can be examined after the fact. As in the relationship between a recipe and the work of cooking, we have assured that what we do leads to an edible end product or to anything looking like the contents of the photographs in the cookbook. We consider the actual work in terms of the structure relating work and account. In the present situation, this might have been 'doing [identifying common attributes of cubes, spheres, cones, cylinders, and pyramids' (see Fig. 8.2, C7, column 2, bullet 2). Whereas in some instances this work might be straightforward, in the fragment presented above it turns out to be a little bit more complicated. In fact, it is precisely this complication that reveals the real work involved obučenie, which has become invisible in those instances where students, such as Alicia, reply in a way that the subsequent turn treats as acceptable and accepted. Most importantly, perhaps, the relational approach that arises from the later, Spinozist-Marxian Vygotsky should inspire us not to make the forgone conclusion that any curriculum materials could bring about – i.e. 'cause' or 'mediate' – student learning such that whatever is planned as the outcome is actually produced. Considering the living mathematics curriculum, *obučenie*, as drama allows us to escape from the tendency to approach the educational enterprise deterministically.

Obučenie as Drama

In the fragmentary text 'Concrete Human Psychology', where Vygotsky articulates sociogenesis as the process of cultural production and reproduction of higher psychological functions, he also envisions a psychology in terms of drama. He notes specifically that psychological functions come to be distributed into the joint behavior of two or more people, 'into a small drama'.[25] We can immediately see the pertinence of his statement to the lesson fragment described and analyzed above. Thus, with respect to Mrs. W., analytically separating her for an instant, we can say

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[25] Lev S. Vygotsky, 'Concrete Human Psychology', *Soviet Psychology* 27 no. 2 (1989), 58.

that in her individual behavior, the coordination of the work of identifying and moving along an edge together with the account exists as one. But, in the *obučenie* activity that she jointly produces with Thomas, that behavior which was fused into one, now is divided anew into two: She verbalizes the account and Thomas acts. Of course the event is dramatic because there is a transactional relation, but more importantly, because for a long time in its course, the coordination is not successful, or rather, the account (instruction) does not go with the action observed. It is only when the small drama has come to its end that it also has a happy ending – at least, to some extent. This is also where Vygotsky departs from his beloved Spinoza, such as when he notes that 'language injects into consciousness not an idyll but drama, even tragedy (an <u>insoluble</u> one). In general, the life of consciousness – unlike the life of the organism – (this is what places the life of the organism outside organic life) – is not an idyll, not Spinoza's serenity, but tragedy: *amor fati*'.[26] Vygotsky's notion of 'organic life' pertains to our organic bodies, movements, and all the things that do not require verbal consciousness. Language itself is dramatic, not when considered as semantic system, but in actual, *dialogical* use. It is also dramatic in the sense that a distinction becomes apparent between organic life and that which doubles organic life, primary *pereživanie*, in ways no less organic, doubled *pereživanie*. The basic structure of higher psychological functions, such as 'the control of one's own body', is social, and, therefore, 'can be most foully developed in the form of *drama*'.[27] When psychology is practiced in terms of drama, when 'psychology = drama', then it is also humanized.

Vygotsky never actually got to work out and experiment with the drama approach to psychology, though at least one interpreter of his texts outlines a number of connections to other terms in Vygotsky's vocabulary that are associated with a theatrical conception of psychology.[28] Accordingly, and with a particular orientation to development in children, the encounter of a child with specific always cultural-historically contingent demands creates tensions, conflicts, that is, contradictions. In the dialectical materialist discourse, contradiction is often stated to be a 'moving force'. But such statements often are read and taken up as if 'contradiction' meant 'logical contradiction' – e.g. the one in chapter 4 where Melissa, Sylvia, and Jane make different models of the same mystery object. A logical contradiction is not a moving force in itself – there initially are no implications of having produced different models within the group of the three girls. But when 'inner contradiction' is seen temporally, as a characteristic of an event, then it pertains to the differences between its earlier and later parts in which initially differing manifestations come to work themselves out. Contradiction then simply refers to the plurality of a unitary phenomenon. Just as the self-movement of drama is worked out within the drama itself – as per Spinoza, a substance cannot be affected by some-

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[26] Lev S. Vygotsky in Ekaterina Iu. Zavershneva, 'The Vygotsky Family Archive: New Findings', *Journal of Russian and East European Psychology* 48 no. 1 (2010), 54.

[27] Vygotsky, 'Concrete Human Psychology', 59.

[28] Nikolai Veresov, 'Zone of Proximal Development (ZPD): The Hidden Dimension?' In *Språk som kultur – betydningar i tid och rum,* edited by Anna-Lena Østern and Ria Heilä-Ylikallio (Vasa: Åbo Akademi, 2004), 13–30.

thing other than itself so that there can only be self-change of the system as a whole – arising from the transactional relations within it, so *obučenie* in the preceding fragment works itself out as the self-integrating movement of the mathematics lesson as a whole.

When there are apparent contradictions from the perspectives of adults – such as when Thomas moves his finger down the center of the cube rather than as instructed along an edge – it is easier to see that developmental events involve 'dramatic collisions' or 'dramatic contradictions-collisions'. But reducing drama to social conflict does not appear to be a productive way to go because 'drama' is more than what we consider 'dramatic' in common parlance, where it connotes something animated, or outstanding and out of the ordinary. Instead, the notion of drama is at the level of the person what the dialogue is at the level of language. It emphasizes the transactional relations of whole people involved in the production of life that occurs with the production for and of human needs in the course of which humans produce themselves. Thus, for Georges Politzer, whose texts inspired Vygotsky to think about psychology in terms of drama, the actual or failed marriage between some Mr. … and Ms. … is a form of drama.[29] Our whole life takes the form of drama; and when we are asked to talk about something, it tends to be in the form of accounts of relations with others and the natural world. Our narratives, as our experiences, are dramatic in nature. The conception of life in terms of thought – as this occurs in all approaches to psychology – is in fact *animistic*, where thought – as mind, spirit, or soul – comes to animate the material body. *Animistic* psychology, where thinking is investigated independently of the real lives of people in the fullness of their involvement in everyday affairs, therefore, is a manifestation of Cartesianism and the whole Western (i.e. Greco-Roman) philosophical tradition termed metaphysics.

The lesson fragment featured above exhibits how drama arises out of drama. Thus, Thomas initially got to go to the front of the classroom after he had raised his hand in reply to a teacher query concerning 'what else' there is to say about edges. Out of this new situation then arises yet another one, when it turns out that there is a divergence with respect to the orientation of the child (treated to be one toward the vertices) and the orientation of the teacher (apparently toward the edges). There is then an issue not only of producing an adjective describing the different forms that edges can take and the actual identification of some physical feature to go with the term 'edge'. The drama is temporarily resolved, after also involving some other actors, when Thomas identifies to the satisfaction of Mrs. W. one of the edges on the cylinder, and then accepts the invitation to sit down. The lesson, the living curriculum, here produces and reproduces a part of itself in a school-specific event denoted as *obučenie*.

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[29] Georges Politzer, 'Les fondements de la psychologie: psychologie mythologique et psychologie scientifique', *La Revue de la Psychologie Concrète* 1 (1929), 28.

9

Mathematics in the Drama of Life

> Cette *vie humaine* constitue … un drame [This *human life* constitutes … a drama].[1]

Throughout this book, I argue for pursuing an understanding of real behavior – mathematics as it shows up and is brought to bear in the different situations in which we find ourselves – rather than some ideal essence. This mathematics is not decontextualized, as in schools where it is practiced at particular times (hours) of the day. The *mathematics of mathematics* practically exists in our world as an ensemble of societal relations, synecdochically manifested as the ideal (suprasensible) and universal of sensible objects. For Vygotsky, the interests, needs, and inclinations of the whole person are the most important requirements for the appearance of particular behaviors and associated forms of thinking.[2] Because inclinations, needs, and interests stimulate activity, any change in needs or type of activity also will change the behavior in some particular setting. On a walk along the beach, an astronomer might talk about the beauty of a sunrise or sunset, about the movement of the sun across the sky, rather than talking about the rotation of the earth that makes the sun move only apparently. A casual observer overhearing the conversation will not pick out from the conversation that there is an astronomer. Knowing that the person also is an astronomy professor at the local university does not assist one iota in understanding the unfolding conversation and relation within the group of people overheard. All the mathematics and all the scientific conceptions characteristic of the work on the job appear to be irrelevant to the event on the beach. These needs and interests in part have a temporary nature, driven by and characteristic of the objective activity and the environment that it

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[1] Georges Politzer, 'Les fondements de la psychologie: psychologie mythologique et psychologie scientifique', *La Revue de la Psychologie Concrète* 1 (1929), 26.

[2] This triad appears in many places, including Lev S. Vygotsky, *The Collected Works of L. S. Vygotsky Volume 5: Child Psychology* (New York: Springer, 1998), chap. 1; and Lev S. Vygotsky, 'Thinking and Speech', in *The Collected Works of L. S. Vygotsky. Vol. 1: Problems of General Psychology* (New York: Springer, 1987), 50.

offers. Whether we observe mathematics and of what kind it will be also depends on the environment – indeed it depends, as seen in chapter 6, on the transactional person–environment relation. The needs and interests are transformative in the sense that they change the horizon, that is, the way in which the objective environment appears in the consciousness of the participants. The needs and interests exist in generalized form as objects/motives of the different societal activities, which produce in general form whatever is required to meet specific needs. By participating in these different activities, we also realize the motives, needs, and interests. Any individual therefore is characterized by the totality of objects/motives realized. In any particular situation, the different psychological functions, each of which has been a social relation first, are part of transactional relations that determine the specificity of the function and the overall hierarchy within the individual. Vygotsky therefore set as task to do '(concrete psychology) to study the different spheres of behavior (professional complex, etc.), the structure and the hierarchy of functions where they relate to and clash with one another'.[3] He knows that 'without the human … *as a whole*, one cannot explain the activity of the human being's apparatus (brain), that … without the person one cannot understand the person's behavior, that psychology must be developed in the concepts of drama, not in the concepts of processes'.[4]

Readers will have little problem to see that much of the research literature is sinning against Vygotsky's recommendations. What students or teachers do tends to be reduced to the consideration of the task, and the behavior then is reduced to the relation of 'concept[ualizat]ions', 'constructions', 'meanings', 'discourse', 'identity', 'positioning', 'intentionality', 'alienation', and so on depending on the particular interests of the authors. Some studies already show how mathematical behavior changes with the situation, often introducing 'boundary' and 'third space' conceptions to provide a rationale to explain the different behaviors observed across settings. Few studies conceive of the changing nature of mathematical behavior in terms of its constitutive relations, such that apprentice electricians are described as not only performing mathematics differently in college and at work but that the descriptions also include their becoming aware of these different performances and their use of the contradiction to constitute what it means to be an electrician journeyperson. The purpose of this chapter therefore is to provide (a) a historical sketch of thinking about specific performances holistically through the lens of the whole person and (b) a sketch of a holistic analysis of mathematical performance in the real everyday world of a fish culturist. The mathematics of mathematics is also that: It includes not only the objects, interests, and pursuits of pure mathematicians but also the mathematical objects and forms of reasoning as people use them in different walks of life for their specific purposes.

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[3] Lev S. Vygotsky, 'Concrete Human Psychology', *Soviet Psychology* 27 no. 2 (1989), 70.
[4] Vygotsky, 'Concrete Human Psychology', 71, emphasis added.

Nothing but the Whole Person

The Spinozist-Marxian turn of the late Vygotsky also has implications for how we think about a concept that has great currency in the mathematics education discourse: identity. If, as developed in chapter 6, any performance has person and environment characteristics, and if *perez̆ivanie* is a *one-occurrent event*, then there cannot be anything like identity – in the way this notion is used in mathematics such as $x = x$ and $\neg x \neq x$. This is so because there is continuous change, a person is continuously becoming, always only grasping an event after its completion and after it has become *that* event. Who the person is no longer is independent of the environment, just as what the relevant environment is inseparable from the person. The concept of identity is problematic because activity cannot be the result of '*external objectification of ideas*, plans, and conceptions created' in the head of the individual, for this would make it 'in principle impossible to say what was the source of the thought'.[5] There are also discussions about how the individual 'constructs' herself, her identity, her position, and so on. The Spinozist-Marxian position Vygotsky was beginning to take toward the end of his life recognizes the transactional relations in we all are caught up, the social-material world that constitutes our objective environment, which, within our individual consciousness, appears as a horizon. Thus, philosophers have acknowledged for some time that circumstances make persons as much as persons make circumstances, which is very different from the autopoietic narratives that we read so frequently in the literature. As quoted in the preceding chapter, humans make each other rather than making themselves.

   A second radical shift in Vygotsky's position important to the way in which we theorize knowing and learning mathematics resides in the shift from pure cognition – knowing, learning, construction, which focuses on thinking only and, therefore, reifies the mind–body dualism – to the whole person in the fullness of life. When the focus is on thinking and learning, to paraphrase Vygotsky, thinking itself becomes the thinker of thought – a critique that years before *Thinking and Speech* he encountered in G. Politzer's article, who states that in psychology, 'there no longer is the question of a man who has killed another main, but the action of a representation on another representation; of mechanical, dynamical, energy-giving, economical, etc. that exist between psychological phenomena'.[6] This has the result of that thinking is divorced from the experienced fullness of life, from the motives interests, and inclinations of the person who also thinks. In traditional psychology, these human concerns and experiences – which are always part of human stories – are replaced by relations between things.

   Vygotsky also draws from the materialist philosopher and Spinoza specialist Ludwig Feuerbach, from whom Karl Marx took over many aspects, even forms of expression. Feuerbach and Marx had called for a new foundation of philosophy by

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[5] Evald V. Il'enkov, *Dialectical Logic: Essays on its History and Theory* (Moscow: Progress Publishers, 1977), 237.

[6] Politzer, 'Les fondements', 36.

turning to the practical problems of humans and both 'regarded human beings in their empirical social contexts as carriers of the cultural process'.[7] Vygotsky thus finds in Feuerbach's texts a basis for developing a human psychology, especially taking up the point that 'Man thinks, not the I, not reason'.[8] Thus, he writes, 'man controls the activity of his brain from without through stimuli' and later elaborates, 'without the human … as a whole, one cannot explain the activity of the human being's apparatus (brain), that *man regulates or controls his brain, the brain does not control man*' and '*when Politzer says: it is the person who works, not his muscles, he has said everything* that needs to be said'.[9] But Vygotsky follows Marx in taking a historical approach to analyzing the concrete conditions of human behavior that are only abstractions in the works of the materialist philosopher. He therefore suggests developing a psychology of human behavior based on the drama of human relations rather than on mental processes – an idea that we find in the original Politzer article, which states that '*the experience that psychology offers to us* is constituted by *processes* that do not have the form of our everyday actions'.[10] Psychology replaces real human relations, the drama of life, by representations, images, and instincts, characters that we do not encounter as such. The psychology that Politzer and Vygotsky are working toward is a concrete, specifically *human* psychology rather than a psychology of abstractions or a psychology grounded in the biology and the physiology of the brain. In terms of behavior, Vygotsky concludes, there is no overlap, no common approach possible to zoopsychology and human psychology.

   In 1928 or 1929, Vygotsky became familiar with the work of Politzer, a French psychologist and philosopher of Jewish-Hungarian origin. Two works had appeared during those two years: a book entitled *Critique of the Foundations of Psychology* (1928) and the opening article of the first issue of a short-lived journal *La Revue de Psychologie Concrète* (1929) entitled 'The Foundations of Psychology: Mythological Psychology and Scientific Psychology'.[11] In both works, the idea of psychology in terms of drama is developed. In 'Concrete Human Psychology', where Vygotsky states that any higher psychological function was a (concrete) human relation first, he repeatedly references Politzer, noting, for example, that a 'higher process (voluntary attention)' can be unfolded experimentally 'into a *small drama*. See Politzer: *psychology in terms of drama*'.[12] He later notes, 'psychology = drama', 'drama truly is full of *such connections:* the role of passion, niggardli-

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[7] Sidney Hook, *From Hegel to Marx: Studies in the Intellectual Development of Karl Marx* (New York: Columbia University Press, 1994), 272. See also Institut für Marxismus-Leninismus der ZK der SED, 'Vorwort', in *Werke Band 40: Ergänzungsband – Schriften bis 1844 Erster Teil* by Karl Marx and Friedrich Engels (Berlin: Dietz, 1968), v–xxxv.

[8] Ludwig Feuerbach, *Sämtliche Werke, Zweiter Band* (Leipzig: Otto Wigand, 1846), 339.

[9] Vygotsky, 'Concrete Human Psychology', 71.

[10] Georges Politzer, 'Les fondements de la psychologie: psychologie mythologique et psychologie scientifique', *La Revue de la Psychologie Concrète* 1 (1929), 30.

[11] Georges Politzer, *Critique des fondements de la psychologie: la psychologie et la psychanalyse* (Paris: Éditions Rieder, 1928) and Politzer, 'Les fondements'.

[12] Vygotsky, 'Concrete Human Psychology', 58.

ness, and jealousy in a *particular* personality structure', 'the dynamic of the personality is drama'. A drama is always a struggle of *such connections* (duty and feeling passion, etc.). A drama cannot be otherwise, i.e., it is a *clash of systems. Psychology is "humanized"'*.[13] Vygotsky reiterates and augments this take further, noting that 'the dynamics of the personality = drama, *sociogenesis is the one true perspective i.e., mechanisms are created in the environment*'.[14] The higher functions and their relations therefore constitute a form of drama, they are the residues and reflections of the real dramas that a person has lived. This is a core result of Marx's analysis of the ideal (universal), which arises in and from (exchange) real relations between people, where the specific, supersensible exchange-value a thing obtains is shared across society, and therefore is universal.[15] Each person, then, is thought as a participant in one of the many dramas of human life, and personality, qua ensemble of concrete societal relations, is a real life drama.

Opposing idealist philosophies, which had reached their height in the works on cognition of Kant and Hegel, the Spinozist-Marxian approach begins theorizing cognition (thought) with real life. This research is not without presuppositions and premises. But the 'the premises from which we begin are not arbitrary ones, not dogmas, but real premises from which abstraction can only be made in the imagination. They are the real individuals, their activity and the material conditions of their life, those that they find already existing and those produced by their activity. These premises can thus be verified in a purely empirical way'.[16] This is precisely the approach we find in a proposal for how a concrete human psychology should operate. Thus, 'instead of beginning by the enumeration and definition of an ensemble of traditional notions, one has to start instead the analysis of the dramatic facts themselves: of the work, as it exists in the factory and everywhere where people are busy with determined tasks; the occupations as they practice them, etc.'[17] As a result of such a focus on the real life process of real, concrete persons in flesh and blood, we get the opposite of what Marx refers to as history as a collection of dead facts or an imagined action of an imagined subject.[18] In this alternative approach, history arises from the inner dynamic of the events under consideration rather than from finished ideas and concepts that people carry around in their minds in the way that idealism has it.

Just presenting the drama, however, would not make psychology scientific. In the drama of life, human actions are explained by general factors and in superficial, commonsensical ways. Thus, the great novelists (e.g. Dostoevsky) present protagonists who, for example, systematically undo the very happiness that life prepares for them. But the descriptive 'precision does not go further than the presentation itself. On the other hand, we do not see the genesis itself of taste, of misfortune,

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[13] Vygotsky, 'Concrete Human Psychology', 67.

[14] Vygotsky, 'Concrete Human Psychology', 68.

[15] Karl Marx and Friedrich Engels, *Werke Band 23: Das Kapital: Kritik der politischen Ökonomie Erster Band Buch I: Der Produktionsprozeß des Kapitals* (Berlin: Dietz, 1962), 85–98.

[16] Karl Marx and Friedrich Engels, *Werke Band 3* (Berlin: Dietz, 1978), 20.

[17] Politzer, 'Les fondements', 46.

[18] Marx and Engels, *Werke Band 3*, 27.

beginning in and with the singular life of the particular individual'.[19] That is, a scientific approach shows how phenomena of order arise in and through the relational (transactional) work of people and their environment, which includes other people and the language they use. Thus, 'concrete psychology does not put itself in the place of the person; it analyses the drama and explicates the drama by what effectively explicates it in qua drama, that's all'.[20] As a consequence, the dramatic events are not explicated in terms of more fundamental or deeper dramatic elements. Instead, the parts of a drama (lesson fragments used throughout) always involve the person as a whole considered in terms of his or her singular life. We do not need to decompose what a student – Aurélie and Mario in chapter 7 or Thomas in chapter 8 – *thinks* or *feels*, accompanied or not by accounts of what the respective teacher thinks, feels, or said to have thought and felt. Instead, we focus on the concrete joint labor they accomplish qua members to the setting to get their job done and, as necessary, by making this labor and its result visible to one another to ascertain that they 'are on the same page'.

In the fragmentary text on concrete human psychology, Vygotsky repeatedly makes statements about personality, which describes the person as a whole. He draws from the recently published 'Theses on Feuerbach', where Marx and Engels state that the human essence exists in the form of societal relations. For psychology this means that 'personality [is] the totality of societal relations. Higher psychological functions are created in the collective'.[21] These social relations occur across different contexts, societal activities, but are interrelated within the person, are characteristics of personality. Any issue arising for a person within an activity is related to other activities, for Vygotsky, therein following a Marxian re-reading of Feuerbach and Spinoza, considers the person as a whole, with his/her motives, interests, and inclinations. The psychological functions of the person are not fixed, do not exist as a stable structure, but instead '*functions change the hierarchy in different spheres of social life. Their conflict = drama*'.[22] Vygotsky exemplifies his approach where the professional and personal spheres come into conflict in the case of a judge whose wife is 'bad'. When the professional sphere is brought to bear on the situation, the person judges the woman, but when the personal sphere is brought to bear, the emotions (love) dominate over the professional sphere. The actual behavior toward his wife will be the result of the *dramatic* – read transactional – relations between the two hierarchies. The outcomes of the resulting struggle will have to be worked out each time such a situation unfolds and cannot be given in the abstract.

The American pragmatist philosopher John Dewey expresses himself in a similar way when stating that 'we live from birth to death in a world of persons and things which in large measure is what it is because of what has been done and

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[19] Politzer, 'Les fondements', 52.

[20] Politzer, 'Les fondements', 56.

[21] Lev S. Vygotskij, 'Konkretija psixologija čeloveka', in *Psixologija razvitija čeloveka* (Moscow: Eksmo, 2005), 1030.

[22] Vygotsky, 'Concrete Human Psychology', 69; and Vygotskij, 'Konkretija psixologija čeloveka', 1031.

transmitted from previous human activities. When this fact is ignored, experience is treated as if it were something which goes on exclusively inside an individual's body and mind. It ought not to be necessary to say that experience does not occur in a vacuum. There are sources outside an individual which give rise to experience'.[23] The first sentence thematizes the cultural-historical dimension of the contexts in which we live. There is continuity in culture that needs to be theorized as the context in, and condition under, which we experience events. The implications of the failure to make this thematic are topicalized in and by the second sentence: experience would be treated as something that goes on exclusively *inside* the body and mind of a person. That is, experience is *not to be thought of* as happening in the body and mind of a person but as distributed across contexts and time. There are aspects, as the quotation states, that fall outside the boundary that we traditionally make between individual and context that contribute to and shape experience. But these are integral parts of the lives we actually live. What we learn arises in the form of consciousness of the dramatic relations lived, for 'consciousness [Bewußtsein] can never be anything else than conscious being [bewußte Sein], and the being of men is their real life-process'.[24]

Two Case Studies

Over the course of a five-year period, my team and I spent a lot of time in a fish hatchery that is part of the federal Department of Fisheries and Oceans Canada. During that time, I got to know two of the fish culturists particularly well, Mike and Erica, who also had a mentor–mentee relation. My research team not only studied knowing and learning of these individuals but also the history of this hatchery within the 120-year history of fish hatching in that part of the country and the changes in that hatchery and human practices over its existence, and those changes that occurred during the even shorter period of our presence.[25] We got to know our participants not only when they worked but also more closely in away-from-work and home settings.

Mathematical Behavior in the Drama of Life

Erica was one of the fish culturists, responsible, among others, for the entire production of coho salmon, beginning with the taking of eggs and milt from animals that returned to the hatchery until the moment that the hatched and raised salmon

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[23] John Dewey, *Later Works Vol. 13* (Carbondale: Southern Illinois University Press, 2008), 22.

[24] Marx and Engels, *Werke Band 3*, 26.

[25] A full report of the study exists in Wolff-Michael Roth, Yew Jin Lee, and Leanna Boyer, *The Eternal Return: Reproduction and Change in Complex Activity Systems* (Berlin: Lehmanns Media, 2008).
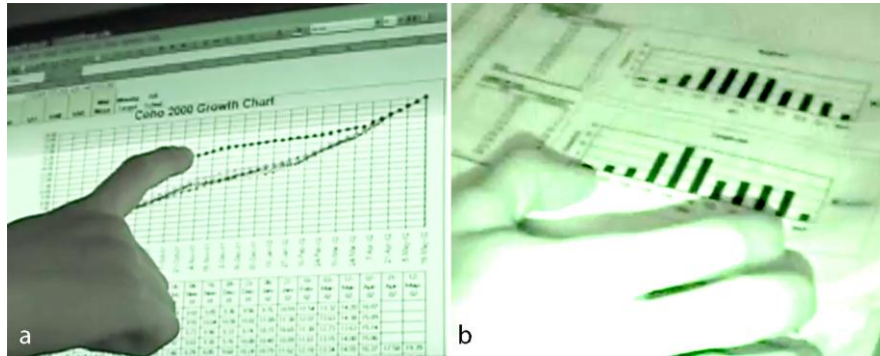
Fig. 9.1 **a** A graph is used to model the actual mean fish weight and compared to a previously established reference (pointed to). **b** The histograms of weights and lengths of a sample of fish, represented in the graph as a single point

are released. After high school, she had attended a business program in a local comprehensive college for two years but then dropped out. Erica recorded all data that she and the temporary workers generate. More than any other fish culturist working in the facility, she liked to 'play [mess] around' with the data by representing computer-based data in different ways or by correlating different pieces of information, actions that increase her understanding of the production of coho salmon smolts. To extend her competencies of using the spreadsheet, she had studied some statistics on her own and she continued finding out new ways of operating it, for example, in the context of making 'neat' displays of data on the monitor. For example, she produced graphs that modeled the growth of the fish in her care in terms of their mean weight. Although a fisheries biologist had established a norm curve before, she did not want her fish to follow that growth curve which had a period (winter) where the fish hardly grew, if it did so at all (Fig. 9.1a). There was a clearly affective relation to the living beings in her care.

When I first met Erica, she already had more than a decade of experience working as a pisciculturist and, by taking secondment opportunities in related sites such as a research laboratory, she had increased her competencies rearing salmon. Erica was positively inclined to personal learning and development ('I like to be progressive'), making every effort to stay up to date. She not only wrote the required reports but also kept personal binders with information that did not enter the bureaucratic channels, or, when she had entered them in the past, they often had been removed from the official records. Her recordings allowed her to consciously reflect on her work practices. She expressed her own change in emotionally positive ways saying, among others, 'I love learning' and 'I am sort of a geek'. Her peers considered her to be a 'very smart' person and recognized in her behavior competencies that made her an outstanding pisciculturist.

Affect was noticeable not only in the way Erica related to the fish but also when it came to the mathematical representation thereof. Thus, for example, I recorded an event during a sampling episode where the data for the weight curves and dis-

tributions (Fig. 9.1b) were collected. At one instance represented in Fragment 9.1, she commented, with apparent disappointment that the weight of a fish was 'way low', and added after a pause, 'darned'. The speech parameters reflected disappointment, consistent with the reported vocal correlates[26] – only about one-eighth (69 dB) to one-sixteenth (65 dB) of the normal speech intensity (79 dB) and, where measurable, the mean pitch had dropped from a normal of 230 Hz to 176 Hz.

Fragment 9.1
01 E: °w*ay* l*ow*° (69.6 dB, 176 Hz)
02   (1.1)
03   °°darn()*ed*°° (65.4 dB)

In another instance, she looked at the condition factor (a non-linear relation of fish weight and fish length), reported it to others as 'two point eight' – where k_c = 1.0 is normal – and with apparent disgust also manifested in the vocal correlates commented, 'OH:: that was a catfish'. In the interjection, the pitch rose from 300 to over 600 Hz only to drop back to about 300 Hz and a mean of 83 dB of speech intensity. Whereas speech intensity dropped back to about normal in the remainder of the phrase, the mean pitch level (356 Hz) remained much higher than normal.

Here, the affective expressions fell together with the consideration of the animals. But there were affective expressions also when the mathematical representations were concerned. Thus, when distributions became available on the computer monitor, she moved the cursor to the upper end of the histogram of weights and directed my attention with an outburst, 'Oh my god, look at them up there, oh ho ho ho'. She then commented, 'I am there, I am so there', by means of which she indicated to be on target (or better) with respect to the ideal fish size according to the curve she established for herself as target (Fig. 9.1a). She finally made a valuative verbal statement that negated a negative value, 'I'm not even concerned', which itself was associated with vocal correlate of happiness. The voice analysis provides evidence of the high level of emotional energy expressed: There was an increasing average pitch level during the outburst, accompanied by high peaks of speech intensity; even during the subsequent speech productions, the average pitch level was considerably above normal pitch (up arrow in turn 02), and the range between the peaks and valleys was very large. In another instance, her helper suggested dropping a data point, which was combined with a strong rejection articulated in the content and prosody of the reply:

Fragment 9.2
01 H: °drop em from distri°[
02 E: [↑*NO* >YA *DO*NT TAKE IT< O::FF

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[26] Klaus R. Scherer, 'Vocal Correlates of Emotion', in *Handbook of Psychophysiology: Emotion and Social Behavior* eds. Hugh L. Wagner and Anthony S. R. Manstead (London: Wiley, 1989), 193.

Erica had dreaded this moment of the year when the fish she had raised for near-ly 18 months now would be released into the wild; and this dread was amplified by the woes in her life more generally. This spring had been especially difficult for her. The hatchery management had announced that Erica would be laid off because of budget constraints. A lengthy strike in her husband's company had led to his temporary layoff. She worried about not being able to afford their new home be-cause of the inability to make mortgage payments, and therefore she began to con-sider selling it. She did not know how to care for the children, who she did not want to suffer because of their economic woes. The future looked uncertain, which worried the normally bubbly Erica a lot. The situation had begun to affect Erica's work. She always had cared for the coho, which she raised from the egg stage to the moment when the fish turned into smolts (at about 20 grams average weight), ready to leave the hatchery and begin their migration to the ocean. She talked about the fish as if they were her children: the moment the fish were released had always been filled with a mixture of sadness, for her 'babies' were leaving, and joy, for her babies had grown up. This spring was different. This time she was on an 'emo-tional roller coaster'. There had been days when she had neglected the coho over her worries about other, non-hatchery-related things; there were days when the fish were not fed despite having been scheduled to do receive food. That is, the changes in her life outside the hatchery associated with negative emotions also were associ-ated with changes in how well she was doing her job, both around the hatchery and in her office. That spring a fellow worker, manager, or ethnographer watching her everyday actions from the outside might have concluded that Erica was over-rated, that she was not as good a fish culturists as she was said to be. Or, knowing of her woes, they might say that the emotions related to her situation affected her work.

The situation also was associated with her mathematical performance, where errors crept in that previously had not been observed. Here, I cite but one instance. She was doing a special program designed to bring back and enhance the salmon population in a creek in the geographical region. It required raising salmon and transferring them to the mouth of the creek just prior to the point in time when the fish would be chemically imprinted in the environment so that they would return to that creek. As always, she collected any pertinent data and prepared a report sub-mitted for the institutional records. It also included tables in which mean weights of the fish over the 11-day period prior to release were kept, together with the type and amount of feed distributed, biomass, and density in the net pen along with en-vironmental variables, such as salinity, water temperature, oxygen levels. For ex-ample, we can see a quite apparent error in the report concerning 'food conver-sion', which is the defined as

$$food\ conversion = \frac{weight\ biomass\ gained}{weight\ of\ food\ dispensed}$$

She reported a food conversion rate of 0.87 (Fig. 9.2). A quick look at the table shows that the biomass increased by 59.45 kg, whereas only 46 kg of feed was dispensed (the final 6 kg of feed was dispensed on the last day of the program

Morts (Nmkd)	Live Balance	MC 1.2 mm	Ewos 1.5 s Transfer	Total Kg Fed:	Fry Wt. (gm)	Biomass (kg fry)	Density (kg / m2)
0	26,666	NF	NF	0.00	**3.77**	100.53	2.79
0	26,666	3.00	1.00	4.00	*3.93*	104.87	2.91
0	26,666	3.00	1.00	4.00	*4.10*	109.39	3.04
0	26,666	3.00	2.00	5.00	*4.28*	114.11	3.17
1	26,665	3.00	2.00	5.00	*4.46*	119.03	3.31
0	26,665	2.50	2.50	5.00	*4.66*	124.17	3.45
0	26,665	NF	NF	0.00	*4.86*	129.53	3.60
0	26,665	3.00	3.00	6.00	*5.07*	135.12	3.75
0	26,665	3.00	3.00	6.00	*5.29*	140.95	3.92
1	26,664	2.00	2.00	4.00	*5.51*	147.02	4.08
1	26,663	3.50	3.50	7.00	*5.75*	153.36	4.26
0	26,663	3.00	3.00	6.00	**6.00**	159.98	4.44
3	99.99%	29.00	23.00	52.00			**4.44**
		1.5	1.2	2.6	<Bags Fed		
				$82.68	Feed Costs		
		Food Conversion:		**0.87**			

Fig. 9.2 Table with data on a program of enhancing the salmon population of a creek away from the hatchery

when the final weights were measured). That is, there was more growth reported than actual food dispensed, which corresponds to a conversion rate greater than 1. But this is not possible, as Erica well knows. The cell contains the inverse of the actual value for the conversation rate, and mistakenly includes the feed from the same day when the fish were last fed. It also does not take into account the fact that on Day 1 the fish did not receive any food and still increased in weight (Fig. 9.2, first row). The same report contained further errors unusual for the work of Erica, such as when the absolute growth rate of the fish was reported to be 20.3%, when inspection of the table shows that the fish weight had grown by 2.23 gram, which, relative to the starting weight, constitutes 59.2%.

In this subsection, we take a historical look at mathematical behavior in the context of the whole life of one pisciculturist. We note on the one hand the manifestation of the affective dimensions of work, both to the organisms that Erica cared for as to the mathematical aspects of representing the fish populations. The case study also shows that the quality of mathematical behavior changed, which was error free in times of stability in the overall life of the person but was associated with blatant errors ('misconception' in the case of the reversal of the numerator and denominator of the food conversion rate) during a time of upheaval, that is, when she confronted layoff while her husband also was laid off. Moreover, we observe mathematical behavior as part of the overall place work played in the life of Erica, who did not just work for an income but who also found herself rewarded by the activity when she was doing it well and when she was successful – independent of whether the hatchery managers acknowledged it or not. The mathematical representations – e.g. the graph of mean fish weight – not only allowed her to (intellect-

tually) understand where the fish population was in terms of its development – whether it was on track for the projected release date – but also was reflected affectively, negatively when it appeared that the population or individuals were off the projected track, and positively when she was 'dead on'. The constitutive relation between the intellectual and affective dimensions, each a manifestation of the larger whole, was quite apparent in the consideration of the results of a data collection episode, when she stated in a contradictory manner both 'I am so there, I don't even have to be concerned' and 'I am lying, because I am always concerned'. This concern was expressed not only intellectually, by means of the contents of the words, but also affectively, in material form, such as in prosody. This is why Vygotsky considered the study of speaking central to the resolution of the psychophysical (mind–body) problem in psychology: speaking, thinking, and emoting are integral to the overall movement of an event (activity) and cannot be understood independently thereof. This overall unity cannot be reconstructed by any combination of the different manifestations – intellect, affect, or deed. Only whole units, such as the category *pereživanie* capture all of these dimensions as manifestations of an overarching whole – which Vygotsky unfortunately named, in the case of language, *značenie slova* generally translated as 'word meaning' (rather than 'signification', 'value', or 'function').

Between Affection and Disaffection

During the period of our research, Mike might have easily been described as an 'unmotivated' employee. He arrived on the job an hour or more earlier than others, which allowed him to avoid 'facing management'. In the afternoon, he left as soon as he 'had put in his time'. Working in the hatchery during the years we spent there provided him with a stable income, whereas it was away from work that he realized his interest: he built his home, grew vegetables that he sold at the local weekend market, to go traveling, and so on. What we observed was interesting in the light of the fact that in preceding decades, he had been an eager fish culturist. He frequently featured in the newspaper because of the contributions he had made to the fish culture; he had received a national award for his contribution to the area of aquaculture; and he was an eager participant in local and regional conferences where he talked about the advances in individual and collective piscicultural practice. Mathematics was an integral part of his endeavors, especially because any changes in piscicultural practice had to be grounded in empirical evidence. His great contributions to the field arose out of experiments that he had conceived and then developed together with interested scientists from the department. For example, he had developed the idea of raising salmon eggs in moist air rather than in flowing water. The result of the study yielded, among others, a graph in which the dry embryo weights of the eggs in two parts of the two raising methods are displayed (Fig. 9.3). But during the time we spent with him, mathematics at work played a minor role in his life, such as when he determined the total number of
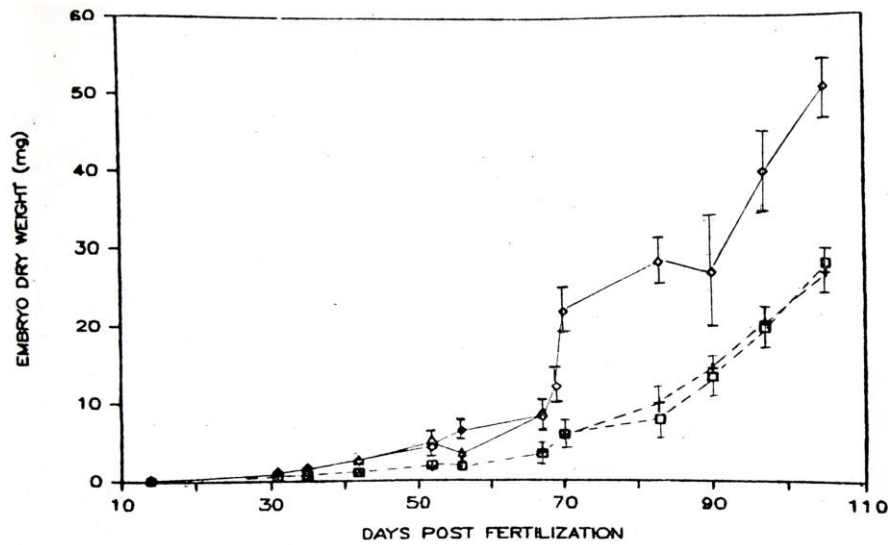
Figure 5. Dry embryo weights of Robertson Creek 1986 brood coho eggs incubated in the moist air incubator and in Heath trays. Left Side - ◊ ; Right Side - △ ; Upper Tray - □ ; Lower Tray - + ; with 95% confidence limits.

Fig. 9.3 A graph used to represent findings from the experiment comparing moist air incubation with regular incubation

eggs culled – from a given volume of all fish eggs, 100 mL were taken and all eggs were counted so that the total number of eggs was calculated to be

$$number\ of\ eggs = volume\ \ L\ \ x\ 10\ x\ number\ of\ eggs\ per\ 100\ mL$$

Attributing motivation or lack thereof to him to the two managers of the facility, with whom he was in conflict, or to the lack of funding in the salmon enhancement program would be abstracting from the complexity of the situation. In an interesting turn of events, following our intensive study Mike got himself involved more fully when Erica returned once a temporary position came open, providing him with an opportunity to work closely with her. Thus, any current needs and interests arose from a hierarchy across all the societal activities in which he participated.

Initially, Mike identified with the collective work-related object/motive. Giving the fullest at work came with satisfaction, which doubled when the work was rewarded. At that time, Mike was giving 'three hundred percent' to the organization. Although he had a family and other interests, there was a time when fish hatching was the most important thing in his life. The performance of the hatchery where he worked and his biography are closely connected, for not only did *he* become known as a leading specialist but also the institution became an 'indicator hatch-

ery', that is, a hatchery that was taken as a measure representing an ideal situation. Each time he found something new about the process of raising young salmon, it increased the options in the hatchery more generally for raising a healthy brood and, thereby, for controlling the various aspects that affected the health of the brood from the fertilized egg stage to the point of release. At the same time, while he gave 'three hundred percent', the hatchery also was the most productive and best-known facility in this geographical region – in as far as the number of salmon produced was concerned as much as in the extent to which it generated new pisci-cultural practices. But this is not to say that the hatchery's success could be attributed to his work alone. Instead, the relationship between the work *environment* and the person Mike is better understood in terms of *pereživanie,* so that success and the positive emotions are the outcome and manifestation of the transactional {person | environment} relation.

Over the five years of our research project, Mike merely put in his hours ('I don't have the fire to get down there and really get it going'). He no longer identified with the workplace. That work now took a very different place in a hierarchy of roles, interests, and needs. It was not at all that he was 'unmotivated' but rather that his 'motivation', was oriented otherwise than, for example, increasing the productivity and available knowledge of this hatchery. It was not at all that he did not feel positive about what he was doing. Instead, he always was very proud of what he was doing, took great care in what he was doing, and was proud of the quality with which he was doing it. Although negativity was apparent – with respect to the workplace, its politics, the favoritism, or its lack of support for innovative practices and innovation – there was also positivity. Yet overall, the mathematical practices observable also changed. Whereas he might have been interested in participating in the production of graphs and understanding what they manifested about the brood that he cared for, it was no longer part of what my research team documented during the years we stayed. But it was not as if mathematics had completely disappeared. Rather, his competencies were indeed showing through in the questions and comments he articulated during several occasions when scientists presented their analysis conducted on fish from the hatchery.

It is interesting, however, and of theoretical importance, that in the years after our research had stopped, Mike was again beginning to identify with his work. He developed a renewed interest in experimentation and in the production of piscicultural savoir-faire. He had taken on the task of sponsoring Erica, who, after having lost the position, was rehired and worked at the hatchery for a period of time. He wanted her to learn everything that he had come to know in the course of his 30 years in the organization. In this instance, much as shown in the case of Mario (chapter 7), participation may give rise to the change in emotions even though it is associated with negative emotions. In his *pereživanie*, negative emotions transformed into positive emotion. They did so, of course, not merely in themselves, but also with changes at the environmental pole, including the reappearance of Erica, another eager pisciculturist, and the changes that their working together brought about in the work environment.
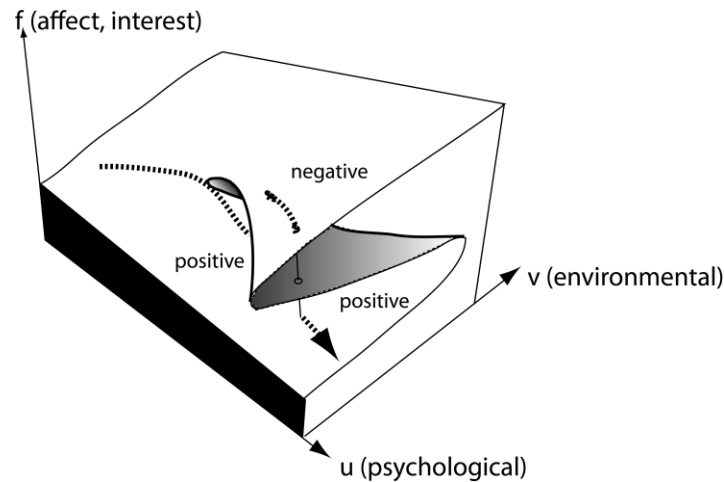
Fig. 9.4 Catastrophe theoretic model relating quantitative changes in personal and environmental characteristics to the qualitative changes in a third variable, here affect. The historical changes in Mike's affect associated with the workplace from positive to negative and positive again would be the result of sudden qualitative transitions that are the result of (sometimes minor) quantitative changes in personal or environmental characteristics

It is not difficult to ascertain the usefulness of what Vygotsky refers to as his schema of interests while presenting the clash of two hierarchies of functions, one in which thought dominates passion, the other where passions dominate thought. The resulting behavior toward the workplace, the institution, is the result of the drama that manifests itself in intellectual and affective terms. That drama is not constant, but a continuous struggle, and, therefore, can change in the course of history, constituting the course of history. In the discussion of interest during children's transitional age, Vygotsky critiques the formal consideration of psychological development for its antigenetic nature.[27] The key problem is that the formal consideration ignores the changes that come about with the changes in the mechanisms of behavior and in the forces driving behavior. In the case discussed here, a historical genetic approach shows changes in a person's behavior over time, the affective relations to a particular (work) environment, and the changes in the mathematical aspects within behavior that can be observed. There is no stability of interest or personality but a moving system that may produce stability or change. And, as we see, it is not that there is a once-and-for-all change in interest and emotion toward work, but that transactional relation, that *pereživanie*, changes.

Pereživanie is the transactional relation of person and environment, together with its refractions in intellectual consciousness and in emotions. Such transactional relations sometimes are characterized as 'dialectical' because of the apparent mutual constitution of the opposing terms – which are but manifestations of an inner unity/identity that appears as this or that. Such relations can be modeled

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[27] Vygotsky, *Collected Works 5*, 3.

mathematically in catastrophe theory, which is a way to express the dialectical *law of the conversion of quantity into quality and vice versa* that was also dear to Vygotsky because it allows distinguishing the continuity of growth and learning from discontinuity of development.[28] In a cusp catastrophe, two variables – here personal characteristics *u* and environmental characteristics *v* – may map onto two qualitatively distinct values of a third entity *f* in part of the region they span, such as positive and negative emotions (or interest and disinterest, affiliation and disaffiliation) (Fig. 9.4). Mathematically, this is expressed as $f = f(u,v)$. In the critical region of the personal and environmental variables, minor changes can lead to a sudden, qualitative transition in the third variable, which, depending on our particular research focus, may be interest, emotion, or affiliation with the workplace. In the non-critical regions, quantitative changes in *u* or *v* are related to quantitative changes in a single value $f(u,v)$.

Mathematical Thinking and Emoting in the Drama of Life

In this chapter, the mathematics used in a workplace is presented in the context of the whole life of the person. Two case studies are used to show that the observable mathematical behavior changes with the changing conditions in the life of the person; and even the importance of mathematics and the interest of the person to employ it may change given all the other events in the person's life. I know from both teaching and research experience that students and adults alike do not tend to speak about other things in their life that turn out, once we do ask, to play a significant role in the drama of life that also has repercussions for their mathematical behavior in any one of the societal activities in which they participate. Mathematics education (research) does not tend to account for all of the things that go on in its participants' lives. In many areas of mathematics education (research) we may observe reductionist tendencies of various kinds. In schools, mathematics is taught in lessons apart from other lessons, in little packages, a given number of lessons per week. There are many studies that show that school mathematics is different from the mathematics students do elsewhere, and success in school mathematics is unrelated to successful use of mathematics elsewhere in life. Other reductions occur when mathematical competencies are investigated using clinical interviews, for example, which do not tend to account for the fact that the behavior of participants is already co-determined by the fact that they participate in research using the special form of clinical interview and not something else. When students are isolated from other students to sit examinations, a further reduction is occurring because behavior is considered with the person disconnected from the kinds of social relations that a mathematician, physicist, electrician, or pisciculturist would be draw-

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[28] For the dialectical laws see Friedrich Engels, 'Dialektik der Natur', in *Karl Marx and Friedrich Engels Werke Band 20* (Berlin: Dietz, 1975), 348; and Lev S. Vygotsky, 'Tool and Sign in the Development of the Child', in *Collected Works Volume 6* (New York: Springer, 1999), 3.

ing on throughout any working day. That is, even though *Thinking and Speech* exhorts its readers in the opening pages to investigate thinking in the full vitality of life and in the context of the motives, interests, and inclinations of the thinking person, mathematical thinking is studied as if it were the thinker of thought. The affective coloring of behavior does not arise from 'mathematics identity'[29], as if humans consisted of a collage of identities related to each school subject. Vygotsky's example of the judge shows that the different spheres relate in dramatic ways, and the outcomes of these dramas cannot be predicted in advance of and independent from their real unfolding in specific life situations.

Vygotsky started a radically different avenue – at least toward the end of his life. He worked on the assumption of the holistic nature of human persons, each of whom also reflects society as a whole. Thus, he writes in 'Concrete Human Psychology', 'To paraphrase Marx: *the psychological* nature of man is the totality of [societal] relations shifted to the inner sphere and having become functions of the personality and forms of its structure'. The text then directly refers to the Sixth Thesis on Feuerbach by stating: 'Marx: man as genus (i.e., the species essence of man); here, the individual'.[30] Any higher psychological function was a social relation, and the '*functions change the hierarchy in different spheres of social life. Their conflict = drama*'.[31] The present chapter pursues such an agenda, where the behavior of a person is investigated across the different spheres of the lives of two individuals working for the same institution. The behavior within any single activity in these lives has to be seen in the context of all the other activities and behaviors of a person, all of which arise from a situation associated with a hierarchy of interests, motives, and inclinations. In the fullness of life, there are constraints, and a person cannot just do as s/he wishes. Mike could just abandon the work in the hatchery because he would have cut himself off from a secure income that he needed to continue all the other aspects of his life, including the construction of a new home. His productions for the garden markets were not sufficiently substantial to sustain him. And finally, he did not dislike being a pisciculturist. In fact, he continued to love doing what he had done for decades. It was only the institutional context generally (lack of funding at the levels of federal department and regional office) and specifically (conflict with management) that was turning him off so that he arranged his work such that he did not have to confront others (beginning and leaving work early, choosing to do those aspects of the hatchery's work that took him away from the facility, fertilizing a lake to support another species of salmon, gathering dead salmon in various creeks and rivers and collecting relevant data, etc.).

In the case of Erica, too, to understand mathematical behavior and the varying manifestations of affect, we need to consider these in the fullness of her life. Just as for Marx religion 'is to be reconstructed from the real condition of man's empirical life and not from his essence', any mathematics and mathematical cognition of the

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[29] See, for example, the review of the literature by Lisa Darragh, 'Identity Research in Mathematics Education', *Educational Studies in Mathematics* (2016). DOI: 10.1007/s10649-016-9696-5

[30] Vygotsky, 'Concrete Human Psychology', 59.

[31] Vygotsky, 'Concrete Human Psychology', 59.

real Erica is to be reconstructed from the real conditions of her empirical life, the simultaneous layoffs that she and her husband experienced, their financial situation and the mortgage on their new home, the particulars of the work in the hatchery, and the various relations to others, including her parents who take part in taking care of the children during the times that school is closed. Thus, if these conditions are such that they generate certain kinds of behaviors, then their emergence as well as their disappearance needs to be approached out of these same real-life conditions that enable and disable any mathematics to be displayed and brought to bear on her work. In Erica's situation, she persisted in doing the mathematical modeling even though the hatchery management never made any use of it and even discouraged the production and use.

Currently, however, most investigations in mathematics education still tend to be concerned with thought, conceptions, and whatever else is extracted from what research participants say – including identity, emotions, self-efficacy and the likes. These are then slated to be the properties of a person under investigations. All these are stable properties said to characterize the person across situations, part of the structure that produces any particular behavior. However, from the Spinozist position, such descriptions are of little use – see, for example, the description of an adequate idea of the circle in chapter 2. Thus, 'the fullest description of the *structure of an organ*', here the thinking body, 'i.e., a description of it in an *inactive* state, however, has no right to present itself as a description, however approximate, of the *function* that the organ performs, as a description of the *real thing* that it does'.[32] We can find the same take in the writings of the late, Spinozist-Marxian Vygotsky when he states that 'any thought has movement. It unfolds. It fulfills a function or resolves some task' and in the process, 'strives to unify, to establish a relationship between one thing and another'.[33] We can learn from this that it makes little sense and will be of little use to study the structures of the mind, whether in the form of conceptions, conceptual frameworks, or schemas. All of these tell us relatively little about the thinking in movement, which always is related to the situation as a whole. Thus, there is not just some task – an examination – that will give a more or less similar result wherever it is completed, including in the isolation of the desk in an examination room. Instead, tasks are solved in relation to the overall activity. For example, apprentice electricians will approach the task of finding how much and where to bend a metal conduit differently whether they are on the job or in the classroom (examination).[34] In the former location, they use a tool with markings, whereas in the latter, they will (try to) use trigonometry. Thinking here is appropriate to the situation as a whole, and, therefore, different across the two settings (workplace, college). This is why the Spinozist-Marxian position takes understanding the mode of action of the thinking body to necessitate considering 'the

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[32] Il'enkov, *Dialectical Logic*, 45.

[33] Lev S. Vygotsky, 'Thinking and Speech', in *The Collected Works of L. S. Vygotsky. Vol. 1: Problems of General Psychology* (New York: Springer, 1987), 250.

[34] Wolff-Michael Roth, 'Rules of Bending, Bending the Rules: The Geometry of Conduit Bending in College and Workplace', *Educational Studies in Mathematics* 86 (2014): 177–192.

mode of its active, causal interaction with other bodies both "thinking" and "non-thinking", and not its inner structure'.[35]

Thinking generally and mathematical thinking specifically is not something that can be studied on its own, for it is not thinking that thinks but the person in the fullness of his/her life; similarly, we cannot study emotions in and toward mathematical behavior, for it is not emotions that emote. Instead, we always find ourselves involved in some kind of project in the context of all other projects that make our lives. Thinking and emoting are part and parcel of these projects; they play important functions within the larger project. Thinking and emoting cannot be abstracted from the project because, as parts, they reflect the whole in the service of which they function. Take away the project, then there is no longer a beginning to which thinking and emoting respond, and there is no longer an ending, which thinking and emoting contribute to realize. Emotions have as one of their special functions the valuation of the probability of success between the actions and the motive of the activity.[36] But that motive is situated within a hierarchy of motives, which, as seen in the case of Mike, changes in the course of life and as the result of activity itself. Even in the case of adults, where personality tends to be thought of as a relatively stable feature, changes in the hierarchy of object/motives continuously occur, which has the consequence that we observe different forms and intensities of thinking and emoting that occur within mathematical behavior oriented toward the realization of the goal at hand. Thus, when the conditions are such that the work no longer is on top of a person's hierarchy and no longer giving 'three hundred percent' – which is realized among others by going out of one's way to create representations that prove the advantages of a particular practice – the observed mathematical practices also change. This is important because our interest lies in understanding *actual practices*, not what we can attribute to the person Mike after disconnecting him from everything else in his life.

Vygotsky started orienting us toward considering the whole person, and considered it in terms of all the real soci(et)al relations an individual has and has had with other persons. *Personality* is that category that therefore captures both the societal characteristics and the personal characteristics, which arise from the specificity of affect to *pereživanie* to the real, lived person–environment relations, where 'environment' includes materials, tools, language, and others. Once we consider persons as irreducible wholes, we see multiple object/motives at work in the lives of Erica and Mike, as in the live of anyone else we care to study. Understanding how these different object/motives relate to each other (hierarchically) leads to uncovering the real determinations of the ways in which Mike and Erica participated (high at home, in farming; low in the hatchery). In fact, it is the very relation of the different object/motives – and, thereby, the activities in which a person participates in the course of his/her total life activity – that defines his/her personality from a cultural-historical perspective. Thus, the 'hierarchies of activity are engendered by

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[35] Il'enkov, *Dialectical Logic*, 45.

[36] Alexei N. Leont'ev, *Activity, Consciousness, and Personality* (Englewood Cliffs: Prentice-Hall, 1978), 120.

their own development, and it is they that form the nucleus of the personality'.[37] The 'knots' that link the diverse activities in which a person participates 'are tied not by the action of biological or spiritual forces of the subject which lie within him but by that system of relationships into which the subject enters'[38], that is, society as a whole. Because society consists of a network of activities, established as the result of a progressive division of labor and the exchanges between the systems of activities, personality, this system of relations, is not the result of the subject. The individual personality, therefore, is the result of the actions of the subject that result in a hierarchically organized knot-work of a set of collective object/motives in the context of the totality of societal relations in which the subject takes part. The individual, here Erica or Mike or anyone else that we care about, is understood as a hierarchy of societal object/motives. This means that if a person appears to be unmotivated, this is merely so because the particular object/motive – e.g. mathematics or fish hatching or mathematics in fish hatching – is very low, vanishing, or inexistent in the hierarchy of motives that characterize and constitute the individual.

We see in both case studies that mathematical behavior in piscicultural practice was tied to the place of work within total hierarchy of needs and interests characterizing the person. Thus, even though Mike was actively involved in the production and transformation of mathematical data to document the usefulness of the practice of moist air incubation, during the five years that I spent in the hatchery, no such behavior could be observed. However, after the project was completed and with the return of Erica, he was cooperating closely with her, including the production of mathematical representations.

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[37] Leont'ev, *Consciousness,* 114.
[38] Leont'ev, *Consciousness*, 114.

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Overcoming Dualisms

> 'If we enrich Vygotsky's ideas with Il'enkov's basic postulates, modem psychology and pedagogy will take a considerable step forward in study of the genesis and development of consciousness of the individual subject of activity'.[1]

In this book, I offer up a way of understanding mathematical learning and development from a non-dualistic perspective the outlines of which L. S. Vygotsky began articulating late in his short life as an alternative to everything that he had done up to that point. This alternative was grounded in his reading of B. Spinoza through the philosophical lens of dialectical materialism first articulated in the works of Karl Marx and Friedrich Engels. Spinoza took Nature, Life, as one substance that manifested itself in the form of two mutually exclusive attributes: Extension (body) and Thought. He notes that there could not be a cause–effect relation between such phenomena because they are external to each other. Once we commit to thinking in terms of the attribute Thought, we are stuck with it. In modern parlance, when we begin with Thought, everything, the whole life, is explained in terms of thought – we operate like idealists. On the other hand, if we begin with the material body, a manifestation of Extension, we equally are stuck with it. The best we can do is biology, not psychology, as Vygotsky notes. We are operating like 'vulgar materialists', who attempt to 'identify thought with the material processes that take place *within* the thinking body (head, brain tissue)'.[2] That is, we cannot get from a materialist account – such as in embodiment and enactivist theories – to thought, just as we cannot get from idealist accounts to the bodily material of life. Both the biological and the interpretive (intellectualist) approaches to psychology are ill-fated endeavors from their beginnings. Importantly, the incompatibility of Thought and Extension (body) cannot be overcome by means of a magical

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[1] Vasily V. Davydov, 'L. S. Vygotsky and Reform of Today's School', *Journal of Russian and East European Psychology* 36 no. 4 (1998), 92.

[2] Evald V. Il'enkov, *Dialectical Logic: Essays on its History and Theory* (Moscow: Progress Publishers, 1977), 35.

trick from Hegel's dialectical toolbox. As the American pragmatist philosopher John Dewey realized, 'What has been completely divided in philosophical discourse into man *and* the world, inner *and* outer, self *and* not-self, subject *and* object, individual *and* social, private *and* public, etc., are in actuality parties in life-transactions'; and he then concludes, 'The philosophical "problem" of how to get them together is artificial. On the basis of fact, it needs to be replaced by consideration of the conditions under which they occur as *distinctions*, and of the special uses served by the distinctions'.[3] On this point, Mikhail Bakhtin notes, 'The actually performed act – not from the aspect of its content, but in its very performance – somehow knows, somehow possesses the unitary and once-occurrent being of life; it orients itself within that being, and it does so, moreover, in its entirety – both in its content-aspect and in its actual, unique factuality'.[4]

Concerning the age-old problem of psychology that continues to plague the field to the present day – e.g. in the form of constructivism and its assumption of the autopoiesis of the mind – Vygotsky wrote in his personal notes that the body–mind (i.e., psychophysical) problem expresses itself in the way that the relationship between thinking and speaking is theorized. For him, resolving the issue of speaking and thinking would be a first step in overcoming the dualism that plagued psychology in his times and that continues to plague not only psychology but also mathematics education to the present day. The common way of theorizing this relation is marked by the fact that mathematical knowing is theorized in terms of mental frameworks, schemas, or concept(ion)s; sometimes investigators think they get out of this dilemma when they theorize behavior in terms of practices. But everything else these researchers do gets them back into the individualism of traditional psychology and the associated dualism. When students speak, their words are treated as external expressions of their inner thoughts. Any interview study, whatever the mathematical phenomenon of interest, is based on this idea that researchers can find in the talk what the student is thinking. In the late Vygotsky's Spinozist-Marxian take, thinking and speaking are two lines of development – G. Deleuze and F. Guattari would say, *lines of flight* – that intertwine and mutually shape each other. In the Spinozist-Marxian approach Vygotsky was taking, thinking is characterized not by the body's own structure, as it is in enactivist and embodiment approaches, but according to the structured and situated ensemble of all other things, including the structures and locations of its own body. As a result, thought becomes itself in speaking rather than pre-exists the act of speaking. In this book, I expand the late Vygotsky's interests by focusing on thinking and communicating, which includes gesturing, bodily positioning, and other forms of moving that are required for expressions to be intelligible.

Vygotsky, following others – including the philosophers Ludwig Feuerbach, Karl Marx, and Georges Politzer – is critical of the abstractions psychologists make. Having abstracted psychological phenomena from life, they subsequently

[3] John Dewey, and Arthur F. Bentley, 'Knowing and the Known' in *Useful Procedures of Inquiry* edited by Rollo Handy and E. C. Harwood (Great Barrington: Behavioral Research Council, 1999), 187.

[4] Mikhail M. Bakhtin, *Toward a Philosophy of the Act* (Austin: University of Texas Press, 1993), 28.

seek to re-establish a unit of body and mind that they had previously destroyed. Why not follow the Georges Politzer text that Vygotsky repeatedly refers to, which questions why researchers have to move away from the real drama of life to abstractions that subsequently need to be re-embodied? 'Of what use is it for me to say, for example, after having stated that I work better on white rather than yellow paper, that I write more easily with a heavy rather than light penholder, that within me there is I don't know what internal experience where the lightness and difficulty are lived, like nothing else in the world is lived?'[5] The representations and schemas of classical psychology lead to the disappearance of the reality we live every day. This reality is not a mere appearance. It is the very substance that everything and anything in our consciousness is built upon. We are therefore better off giving a pass on the generalities that preoccupy traditional psychology and concern ourselves with the drama of real life and the events we experience in more or less dramatic ways.

In Vygotsky's view, even interpretive approaches to psychology are missing what a concrete human psychology is to elaborate, where '*the psychological fact always is a segment of a particular individual's life*'.[6] In contrast, 'the holistic structures as these are elaborated for example by *geisteswissenschaftliche* [cultural science] psychology of Spranger give us the feel of formalism. The "vital forms" of Spranger do not get at the *hic et nunc* [here and now] of the drama; they uniquely relate to the great themes of drama in general'.[7] According to Vygotsky, 'descriptive psychology is *really* developing into fiction'.[8] It does so because it reproduces the mind–body dichotomy that psychology needs to overcome. Only when psychology becomes 'concrete human psychology' does it deal with the concrete things in the life of every single individual, who recognizes the relevance of psychology in and to his/her life. There no longer is a gap between psychological theory and the reality of the classroom life in the way that the teacher experiences and conceives it.

The upshot of these considerations is that we overcome dualism only when, from the outset, we recognize Nature (Life) as manifesting itself in contradictory ways, as Thought and as Extension and therefore begin analysis with a plural singular (One). There already is precedence for such an approach in quantum physics, where a system is modeled by means of equations that have no physical equivalent in themselves. However, when the system is observed, requiring an operation, one or another state is seen – such as in the well-known Schrödinger paradox of the cat in the box that turns out to be dead or alive. In the same way, the *thinking body*

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[5] Georges Politzer, 'Les fondements de la psychologie: psychologie mythologique et psychologie scientifique', *La Revue de la Psychologie Concrète* 1 (1929), 38.

[6] Politzer, 'Les fondements', 43.

[7] Politzer, 'Les fondements', 44. Eduard Spranger was a student of Wilhelm Dilthey, a scholar who inspired qualitative and interpretive methods in psychology, education, and other social (cultural) sciences.

[8] Lev S. Vygotsky, 'The Historical Meaning of the Crisis in Psychology: A Methodological Investigation', in *The Collected Works of L. S. Vygotsky, Volume 3, Problems of the Theory and History of Psychology* (New York: Springer, 1997), 336.

does not exist in the form of some composite of body and thought, though, each time we look, it manifests itself in terms of bodily (physical) and psychological forms (non-physical, ideal, affective).

Vygotsky did not live long enough to enter the 'Promised Land' in which dualism is overcome in psychology, a land that he was seeing from afar but that he, likening himself to Moses, never was allowed to enter. We know that he considered us as taking the 'regal road' to overcoming the problem when we address the sensible–supersensible character of speech, that is, the relationship between thinking and speaking (as physical act). We also know, from the final paragraph of the last book he prepared for publication, *Thinking and Speech*, that he began considering language as historical-formed consciousness articulated by the speaker for others and, therefore, for him-/herself. Consciousness, however, is associated with and a characteristic of human society. This is why in Spinozist-Marxian philosophy the term consciousness is always modified by the adjective societal [gesellschaftlich, obščestennyj] rather than the adjective social [sozial, social'nyj]. The same is the case for the universal, which, in the field of mathematics, practitioners recognize as the mathematics of mathematics. Mathematical universals include generalizations, such as the linear function $C(n) = n \cdot a + b$ that also models the content C of a piggybank in which, beginning with a base amount of b dollars, the same amount of money \$a is deposited each week (n = number of weeks). The ideal of mathematics also includes topological structures, imaginary numbers, regularities in regular numbers, and so on. The ideal and universality are directly linked, as per Marx's analysis, to societal relations.

Marx's analysis in itself shows, using the example of (exchange-) value, how universals (the ideal) emerge in the cultural-historical evolution of exchange relations. Within-group exchanges only lead to particulars – today we would say 'local knowledge', 'situated knowledge'. Only when, with increasing population size, there are life-supporting exchanges of commodities across different societal activities that universals emerge. The exchange relations occur at the level of the entire society and, therefore, despite also being social, are *societal* (characteristic of society as a whole). He draws on etymology, noting that the German noun 'the general', 'das Allgemeine', referred to the land common to all, whereas the particular, 'das Besondre', was a parcel of land that had been singled out. Evald Il'enkov, too, draws on the etymology of his Russian language according to which 'the universal', *obšče*e, is related to society, *obšče*stvo, so that everything related to and specific to society warrants the adjective societal [*obšče*stvennyj]. That is, mathematical truths, the ideal, are societal in nature and practically exist as societal relations, that is, relations that are specific to the human species and which therefore distinguish it from other animal species. The *mathematics of mathematics* thus exists in the form of societal relations, which are reflected in relations between objects.

This way of theorizing has two important consequences. First, mathematical truths, the ideal, exist in and *as* an ensemble of societal relation. Second, because universals exist *as* an ensemble of societal relations, they are not abstract in the common sense of the word, that is, they are not purely ideal. Instead, the ideal of mathematics is completely material, realizing itself in the material relations be-

tween real people. This, then, allows us to distinguish different kinds of relations in the mathematics classroom. When students are asked to work in groups, where they are said to 'socially construct' 'knowledge', concepts, objects, or signs may obtain agreed-upon exchange-value. But the exchange-value – e.g. the function of a word or words – is likely local rather than universal. That is, although students might agree upon using words, objects, signs and so on in agreed-upon ways, these ways do not have to be, and in most cases are not, universal. It is precisely here that the relationship with the teacher is so important, because in her (his) behavior, what once was a societal (i.e. universally valid) relation between people and exchanged thing already has become *One*. In the relation with students, this One, this universal, may again divide itself into two, that is, play itself out as a drama involving teacher and students. Even though the relationship involving the teacher is a social one, it is de facto more: it is a societal one, because in the joint behavior the mathematics of mathematics becomes visible to observers and participants alike. The mathematics of mathematics, the mathematical ideal, therefore is not something in a metaphysical world. Instead, it is here in this real world, observable, in the material relations of people, themselves mirrored in the relationship of things (mathematics words, mathematical signs). The condition for the existence of mathematical consciousness is the presence of what comes to be marked as a specifically mathematical (human) objects as a 'world of things created by man for man, and, therefore, things whose forms are reified forms of human activity (labour), and certainly not the forms naturally inherent in them'.[9]

There is an immediate upshot from the last statement: things and artifacts are not cultural in themselves, that is, they are not containers of cultural forms. Instead, their form is given in, and reflects, the form of human labor as part of which they are used. These things are cultural only in the mutually constitutive relation of human labor and artifacts (produced by humans in the present or past). This is why we cannot expect students, such as Mario and Aurélie (chapter 7), to 'induce' or 'abduce' even such an apparently simple phenomenon as a concrete goblet-and-chip model of a story, and even less to mathematize the series of goblets in the form of additions or multiplications. Even the step from the story of a girl putting $3 into a piggybank each week when the starting amount was $6 is a form of 'abstraction', for at home Mario and Aurélie would have only one piggybank, not six, as in the case of the goblets. The transition might have been easier had the children been taking digital images of the same goblet and, thereby, have obtained an explicitly temporal sequence of states. Each step in the required transformation

story $\rightarrow$ goblets and chips $\rightarrow$ series of additions $\rightarrow$ series of multiplications ($\rightarrow$ generalization in the form $n \cdot 3 + 6$)

constitutes an exchange relation. But exchange relations are relations between people, and the exchange of things mirrors the exchange between people. As a case study of Mario showed, the transformation 'goblet + chips $\rightarrow$ addition' for each of the first four goblets first existed for him *as* and in the form of the relation with the

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[9] Evald V. Il'enkov, 'Dialectics of the Ideal', *Historical Materialism 20* no. 2 (2012), 186.

teacher: as their joint labor, as their joint behavior.[10] It then could be seen in the related transformations from goblet 5 and goblet 6 to the addition form. The 'abstraction' first exists *as* relation, and, therefore, objectively, visible not only to the researchers but to Mario, who then did two such abstractions on his own. What was a relation now is exhibited in and as individual behavior. The mathematical objects, such as '3 + 3 + 3 + 6', '3 x 3 + 6', or 'n·3 + 6', therefore are symbols of *societal* relations.[11] They are not just abstract symbols, but things used and employed in human labor and in human verbal exchanges. They are thus parts of a relation that comes to stand for the relation as a whole: they have synechdochical function.

   The traditional dichotomy of body and mind is overcome in the notion of the *thinking body*. This thinking body has to be understood not in material ways, because doing so would reduce the phenomenon to a mode of Extension and thus constitute a biological reduction. The thinking body does not act upon thought, that is, action is not externalized or materialized thought because the very action of the thinking body *is* thought. In other words – thereby sharply contrasting enactivist accounts whereby a particular hand-arm movement ('gesture') is the realization of a schema – 'thinking does not evoke a spatially expressed change in a body but exists through it (or within it) and vice versa'.[12] When things outside of the body (e.g. vibrating air molecules) affect it somehow – an effect manifesting itself intellectually, affectively – and thereby bring about a change, that change manifests itself in its mode of action, i.e., in its thinking, What we do with things and how we do it is shaped by those things, which in most instances are not natural things but humanly created ones, artifacts. This is clearly seen in such cases as touch-typing, where the keyboard has shaped the habitual movements of fingers and hands such that we type without having to look for the letters on the keys. Experienced typists can get to the point of typing without thinking of words first – they type as they speak, the words coming from the movements of their fingers and hands as the words are coming from their mouths.[13] Those movements, however, are not expressed in the keys or the keyboard (e.g. consider how many people the type by sight and with two fingers). At the same time, the keys are not expressed in the movements considered in themselves (consider the movements of a person waiting, 'impatiently' hitting the desktop with different fingers). The 'ideal' of expert typing manifests itself only in the concrete praxis of human labor that gives rise to the shape of typing movements and the shape of the keyboard with its keys. In a similar way, the ideal of mathematics, that is, the mathematics of mathematics, arises only in those relations of objects and people that are universal; and these relations, therefore, are societal in nature.

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[10] Wolff-Michael Roth, and Luis Radford, *A Cultural-Historical Perspective on Mathematics Teaching and Learning* (Rotterdam: Sense Publishers, 2011), 47–68.

[11] Il'enkov, *Dialectical Logic*, 271; and Evald V. Il'enkov, *Dialektičeskaja logika: očerki istorii i theorii* (Moscow: Izdatels'stvo politi Izdatels'stvoeskoj literatury, 1984), 177.

[12] Il'enkov, *Dialectical Logic*, 35.

[13] For an analysis of touch-typing see Wolff-Michael Roth and Alfredo Jornet, *Understanding Educational Psychology: A Late-Vygotskian Spinozist Perspective* (Dordrecht: Springer, 2017), 228–230.

With his work, L. S. Vygotsky has laid a path toward overcoming all dualisms including those opposing body and mind, individual and collective, abstract and concrete, particular and universal, and material and (purely) ideal. Some of his work has been extended in the writings of the dialectical materialist philosopher E. V. Il'enkov. This book offers mathematics educators a first articulation of a theory grounded in a Marxist reading and re-articulation of the Spinozist idea of *one substance* that as all the different mutually exclusive manifestations as its attributes. We do have the tools required for overcoming the problems of (radical, social) constructivism. It is up to us to employ the emerging non-dualist theory for the benefit of mathematics teachers and students alike.

Appendix

The following transcription conventions are used where indicated.[1]

Notation	Description
(0.4)	Time without talk, in tenth of seconds
(.)	Period in parentheses marks hearable pause shorter than 0.1 seconds
((*turns*))	Verbs and descriptions in double parentheses are transcriber's comments
((*lH, rH*))	Left and right hand in the transcriber's comments sometimes are abbreviated by 'lH' and 'rH', respectively
⌈flip it⌉ ⌊you have⌋ to flip	Square brackets indicate overlapping talk
so flip it	Grey highlighted text within square brackets indicates the extent of the gesture seen in the offprint presented to the right
°y'know°	Degree signs enclose words spoken with less than normal intensity
°°possibly°°	Double degree signs mark speech almost impossible to hear
jUST	Capital letters were spoken with louder than normal intensity
*fair*ly deep	Italicised letters indicate emphases, stress
:	Colons indicate lengthening of phoneme, about 1/10 of a second per colon
>i look at the<	Angular brackets inward mark faster than normal speech
<what happens>	Angular brackets outward mark slower than normal speech
(?)	Missing words, one word per question mark

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[1] Gail Jefferson, 'Glossary of Transcript Symbols with an Introduction', in *Conversation Analysis: Studies from the first Generation* ed. Gene H. Lerner (Amsterdam: John Benjamins, 2004), 13–31.

(stash?)	Word followed by question mark indicates uncertain hearing
.hh	Noticeable in-breath
hh	Noticeable out-breadth
–,?;.	Punctuation is used to mark movement of pitch (intonation) toward end of utterance, flat, slightly and strongly upward, and slightly and strongly downward, respectively
=	Equal sign indicates that the phonemes of different words are not clearly separated, or latching by a second speaker
↑	Significant jump upward of pitch

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